

A novel adaptive filter design using Lyapunov stability theory

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Abstract: This paper presents a new approach to design an adaptive filter using Lyapunov stability theory. The design procedure is formulated as an inequality constrained optimization problem. Lagrange multiplier theory is used as an optimization tool. Lyapunov stability theory is integrated into the constraint function to satisfy the asymptotic stability of the proposed filtering system. The tracking capability is improved by using a new analytical adaptation gain rate, which has the ability to adaptively adjust itself depending on a sequential tracking square error rate. The fast and robust convergence ability of the proposed algorithm is comparatively examined by simulation examples.

Key words: Adaptive filtering, Lyapunov stability theory, Lagrange multiplier theory, error convergence

1. Introduction

Adaptive filters are widely used in a variety of engineering applications due to their generalized adaptation ability. In adaptive filter design, the filter structure is first selected, which may be a finite impulse response (FIR) or an infinite impulse response (IIR). A cost function of the tracking error between the desired reference signal and the filter output is then defined and minimized in the parameter space. The output of an adaptive filter having recursively updated filter parameters can then follow the desired reference signal [1–4].

Among many optimization techniques for optimal filter design, the gradient descent method is the most widely used one. Although it needs a large number of iterations to reach a neighborhood of the minimum point of the cost function, it can be easily implemented [1,2]. Theoretically, gradient descent-based searching may be trapped at local minima of the cost function surface. Moreover, if the input of an adaptive filter has a large bounded input disturbance, it may not even be possible to find the global minimum point.

In order to avoid the above problems faced by the gradient descent-based techniques, Lyapunov adaptive filtering algorithms were proposed in [5–10]. Apart from our proposed design method, they are all formulated by utilizing the recursive least square (RLS) method [3,5]. Unlike gradient descent-based methods, Lyapunov adaptive filtering algorithms are aimed at searching for a global minimum through the error energy surface by adjusting filter coefficients in the sense of the Lyapunov stability theory. The definition and conditions of the Lyapunov stability theory may easily be found in the literature [11]. The adaptation gain rate is the key parameter for convergence behavior in Lyapunov filters. In [5–8], the Lyapunov function, which is the square of error between the desired reference input and filter output, is defined as $V(k) = e^2(k)$, and the adaptation gain rate is determined as a user-defined constant parameter in all computational processes. In [9,10], the

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authors improved the tracking performance of the Lyapunov filter by using a modified Lyapunov function such as $V(k) = a^k e^2(k)$. The adaptation gain rate in [9,10] is also defined as a constant parameter such as a^k , which is exponentially increasing without any adaptive rule. In both approaches proposed in [5–10], an optimal adaptation gain parameter is found after a great number of trials. In order to overcome this problem, we propose an analytical approach to determine the adaptation gain rate, $a(k)$, which has the ability of adaptively adjusting itself.

In this paper, we further investigate Lyapunov adaptive filtering in an optimization framework. A Lyapunov function candidate, $V(k) = a^k e^2(k)$ with $a > 1$, is primarily defined, and then the filter design procedure is formulated as an inequality constrained optimization problem [12]. The optimization process results in a new adaptation gain rate that can adjust itself adaptively in accordance with the sequential square tracking error rate. Thus, filter parameters are updated to make the Lyapunov energy function negative definite at each iteration, $\Delta V = V(k) - V(k - 1) < 0$. With a negative definite sequential tracking error, the effects of input disturbance can then be eliminated, and the tracking error can asymptotically converge to zero. When the proposed filter output tends to deviate from the desired reference signal, adaptation gain rate $a(k)$ changes abruptly, and then the filter output gets close to the desired reference signal and tracks it. The most attractive feature of the proposed algorithm is the adjustable adaptation gain rate parameter, improving the tracking capability of the adaptive algorithm.

The rest of the paper is organized as follows. In Section 2, the novel adaptive filter design approach is formulated and the self-adjustable adaptation gain rate is obtained. In Section 3, simulation examples are given and the convergence dynamics of the proposed adaptive algorithm with adjustable adaptation gain rate are also comparatively evaluated against two other types of algorithms having constant adaptation gain rates. The study is concluded with Section 4.

2. New adaptive filtering algorithm

Figure 1 shows a block diagram of an adaptive FIR filter structure. An adaptive FIR filter can be described as the convolution sum of the input and the filter’s characteristic products. By using a given sample set, $S = \{(\mathbf{x}_k, d_k)\}_{k=0}^{K-1}$, adaptive FIR filter output is formulated as follows:

$$y(k) = \mathbf{w}^T(k)\mathbf{x}(k) = \sum_{i=0}^{N-1} \mathbf{w}_k(i)\mathbf{x}(k - i), \tag{1}$$

where $\mathbf{w}_k(i)$ is the filter coefficient vector for $i = 0, 1, \dots, N - 1$. Filter coefficient vector $\mathbf{w}(k)$ and input signal vector $\mathbf{x}(k)$ can respectively be defined as $\mathbf{w}(k) = [w_k(0), w_k(1), \dots, w_k(N - 1)]^T$ and $\mathbf{x}(k) = [x(k), x(k - 1), \dots, x(k - N + 1)]^T$, where superscript T denotes transpose.

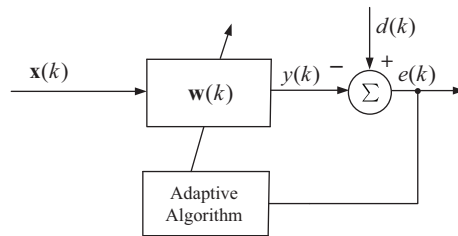


Figure 1. Block diagram of an adaptive FIR filter.

The tracking error between the desired reference signal and the filter output is defined as:

$$e(k) = d(k) - y(k). \quad (2)$$

For design of a Lyapunov adaptive filter, a candidate Lyapunov function $V(k) = a^k e^2(k)$ is first defined and then the following cost function is minimized:

$$\Phi(\mathbf{w}) = \frac{1}{2} \delta \mathbf{w}^T \delta \mathbf{w}, \quad (3)$$

subject to the inequality constraint $\Delta V(k) = V(k) - V(k-1) < 0$, satisfying negative definiteness of the Lyapunov function, which contributes to the asymptotical stability of the filtering algorithm in the sense of Lyapunov. In Eq. (3), the scaling factor $\frac{1}{2}$ is included for convenience of presentation and $\delta \mathbf{w} = \mathbf{w}(k) - \mathbf{w}(k-1)$. This constrained optimization problem proposed here for filter design is indeed a primary problem. The cost function $\Phi(\mathbf{w})$ is a convex function of \mathbf{w} and it can be represented as a linear combination of constrained functions. Accordingly, the proposed inequality constrained optimization problem can be solved using Lagrange multiplier theory [13]. The minimization process is presented as follows:

$$\mathbf{w}^* = \underset{\mathbf{w}(k)}{\text{Argmin}} \left(\frac{1}{2} \delta \mathbf{w}^T \delta \mathbf{w} \right), \quad (4)$$

subject to $(a^k e^2(k) - a^{k-1} e^2(k-1)) < 0$, $a \in \mathfrak{R}^+$. In (4), \mathbf{w}^* represents the vector of the optimal filter coefficients. For solving this problem, we first construct the Lagrangian function as follows:

$$F(\mathbf{w}, \mu) = \frac{1}{2} (\mathbf{w}(k) - \mathbf{w}(k-1))^T (\mathbf{w}(k) - \mathbf{w}(k-1)) + \mu (a^k (d(k) - \mathbf{w}(k) \mathbf{x}(k))^2 - a^{k-1} (d(k-1) - \mathbf{w}(k-1) \mathbf{x}(k-1))^2), \quad (5)$$

where the auxiliary nonnegative variable μ is the Lagrange multiplier. The solution of the inequality constrained optimization problem is determined by minimizing the Lagrangian function $F(\mathbf{w}, \mu)$ with respect to \mathbf{w} and μ . Thus, with differentiation of $F(\mathbf{w}, \mu)$ with respect to \mathbf{w} and μ and setting results equal to zero, the following two conditions of optimality are obtained:

$$\text{Condition 1 : } \frac{\partial F(\mathbf{w}, \mu)}{\partial \mathbf{w}} = 0, \quad (6)$$

$$\text{Condition 2 : } \frac{\partial F(\mathbf{w}, \mu)}{\partial \mu} = 0. \quad (7)$$

Optimality Condition 1 in Eq. (6) yields the following formula after rearrangement of terms:

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \frac{\mathbf{x}(k)}{\|\mathbf{x}(k)\|^2} \left(1 - a^{-k/2} \frac{|e(k-1)|}{|\alpha(k)|} \right) \alpha(k), \quad (8)$$

where $\alpha(k)$ is the a priori estimation error defined as:

$$\alpha(k) = d(k) - \mathbf{w}(k-1)^T \mathbf{x}(k). \quad (9)$$

Similar to those reported in [5–10], the adaptation gain function is defined as:

$$\mathbf{g}(k) = \frac{\mathbf{x}(k)}{\|\mathbf{x}(k)\|^2} \left(1 - a^{-k/2} \frac{|e(k-1)|}{|\alpha(k)|} \right). \quad (10)$$

To avoid singularities in the proposed computational algorithm, sufficiently small slack variable constants are added in the adaptation gain function of Eq. (10) as follows:

$$\mathbf{g}(k) = \frac{\mathbf{x}(k)}{\|\mathbf{x}(k)\|^2 + \lambda} \left(1 - a^{-k/2} \frac{|e(k-1)|}{|\alpha(k)| + \lambda} \right), \quad (11)$$

where $\lambda \ll 1$ and, $\lambda \in \mathfrak{R}^+$. It must be stated here that the tracking error converges to a ball-centered region at the system origin depending on the values of λ [8]. Generally, the smaller λ is, the smaller the error $e(k)$ is. The parameter update law in Eq. (8) can then be simply redefined as below using $\mathbf{g}(k)$ and $\alpha(k)$:

$$\mathbf{w}(k) = \mathbf{w}(k-1) + \mathbf{g}(k)\alpha(k). \quad (12)$$

Let us check and redefine adaptive gain rate a to be used in the computational algorithm benefiting from the negative definiteness of the Lyapunov function, $\Delta V(k) < 0$, as follows:

$$\begin{aligned} \Delta V(k) &= V(k) - V(k-1) \\ &= a^k e^2(k) - a^{k-1} e^2(k-1) \end{aligned} \quad (13)$$

⋮

$$= a^k [\alpha(k) - \alpha(k)\mathbf{g}^T(k)\mathbf{x}(k)]^2 - a^{k-1} e^2(k-1). \quad (14)$$

The substitution of the adaptation gain function $\mathbf{g}(k)$ in Eq. (10) into Eq. (13) gives the following:

$$\begin{aligned} \Delta V(k) &= a^k \left[\alpha(k) - \alpha(k) \frac{\mathbf{x}^T(k)}{\|\mathbf{x}(k)\|^2} \left(1 - a^{-\frac{k}{2}} \frac{|e(k-1)|}{|\alpha(k)|} \right) \mathbf{x}(k) \right]^2 - a^{k-1} e^2(k-1) \\ &\vdots \\ &= e^2(k-1) - a^{k-1} e^2(k-1) \\ &= e^2(k-1)(1 - a^{k-1}) < 0, \text{ for all } k > 1 \text{ and } a > 1. \end{aligned} \quad (15)$$

As seen from Eq. (15), the initial a value should be chosen as $a > 1$ in the computational algorithm for satisfying asymptotic stability in the sense of Lyapunov. Thus, the tracking error can asymptotically converge to zero.

By implementing Optimality Condition 2 in Eq. (7), we get the following adaptive adaptation gain rate formula after rearrangement of terms:

$$a(k) = \left(\frac{e^2(k-1)}{e^2(k)} \right). \quad (16)$$

As seen from Eq. (16), $a(k)$ is indeed the sequential tracking square error rate. In order to satisfy the condition $a(k) > 1$ in the computational algorithm, Eq. (16) needs to be reformulated by adding 1. Consequently, the

new version of the adaptation gain rate is rewritten as follows:

$$a(k) = 1 + \left(\frac{e^2(k-1)}{e^2(k)} \right). \tag{17}$$

The latest version of $a(k)$ in Eq. (17) is used in the proposed computational algorithm for simulations by choosing any arbitrary positive initial value, such as $a(0) > 1$. Thus, the tracking capability is improved by using the new analytical adaptation gain rate $a(k)$, which has the ability to adaptively adjust itself depending on the sequential tracking square error rate. Figure 2 shows the flow diagram of the proposed adaptive filtering algorithm.

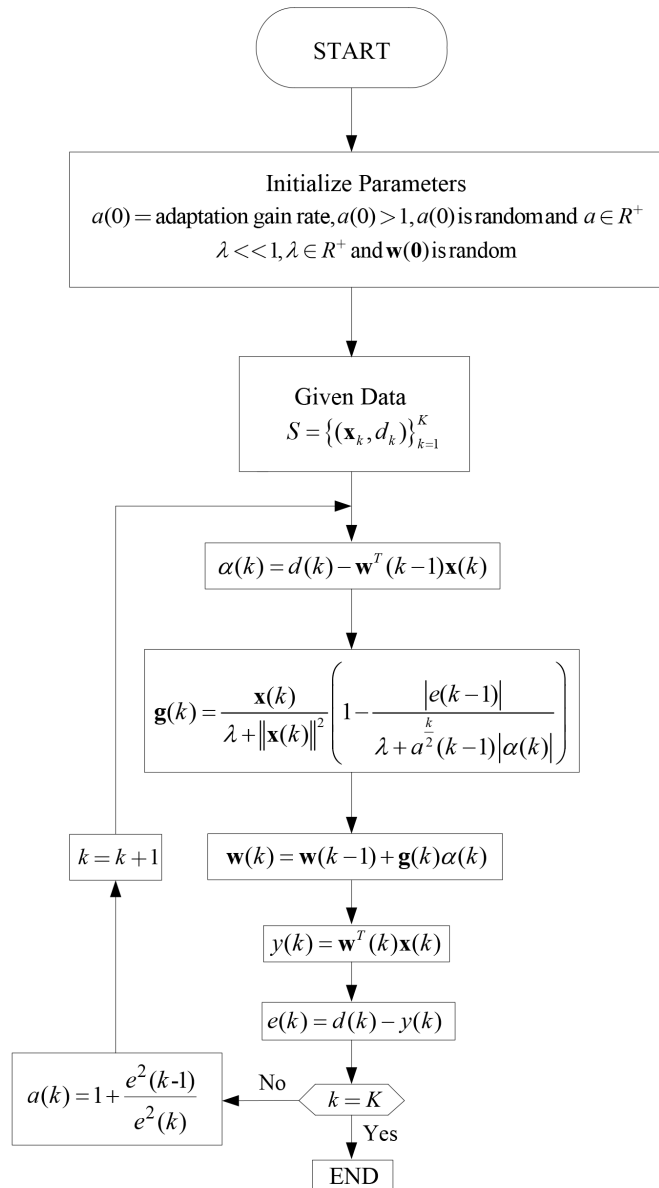


Figure 2. Flow diagram of the proposed Lyapunov adaptive filter algorithm.

3. Simulation results and discussion

In this study, all algorithms are performed using MATLAB software on a PC computer with the speed of 2.53 GHz. In simulations, we have used the following FIR linear combiner for the purpose of evaluating the performance of the proposed novel adaptive algorithm:

$$y(k) = \sum_{i=0}^3 \mathbf{w}_i(k) \mathbf{x}(k-i).$$

In simulation examples, we have used two types of signals. The first is an artificially created signal having abruptly changing points. The second includes well-known benchmark chaotic time series, known as Rossler, Lorenz, and Henon time series signals [14]. The filter order N is selected as 4 and λ is chosen as 10^{-5} for all simulation examples. All the following results are obtained as an average over 30 independent realizations.

3.1. Artificial signal

In Figure 3, the desired reference signal $d(k)$ and noisy input signal $\mathbf{x}(k)$ are shown. Here, input signal $\mathbf{x}(k) = d(k) + n(k)$ is created with an arbitrary noise such as $n(k) = 0.2 + 0.1 * \text{randn}(1)$, which is a nonzero mean disturbance. The filter coefficients $\mathbf{w}_i(k)$ are initialized randomly such as $\mathbf{w}_i(0) = 0.01 * \text{randn}(1)$. a may be chosen and also $a(0)$ may be initialized with any positive real number greater than 1. In these simulation examples, $a = 1.01$ and $a(0) = 1.01$ are randomly chosen.

Figure 4 shows the output tracking of Lyapunov adaptive algorithms for $a = 1.01$ and $a(0) = 1.01$. Although the Lyapunov filters with constant adaptation gain rates a and a^k can track the desired reference signal, their transient responses are not good enough. The overshoots and chattering are easily recognized in the filter outputs when the value of the reference signal is abruptly changed. As seen from Figure 4, the proposed algorithm with its adjustable adaptation gain rate $a(k)$ shows an excellent performance. It has begun to track the desired reference signal a few data points later and it does not deviate from it. In this example, the initial value of $a(k)$ is set to $a(0) = 1.01$ for the purpose of comparison; otherwise, it can randomly be selected as any positive real number that is greater than 1.

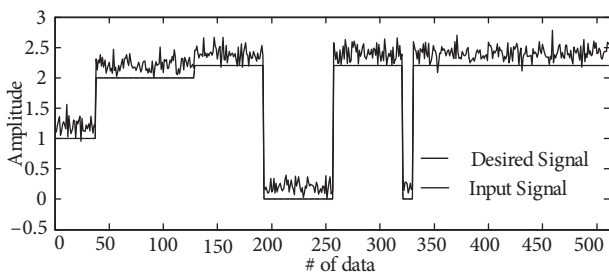


Figure 3. Desired reference signal $d(k)$ and noisy input signal $\mathbf{x}(k)$.

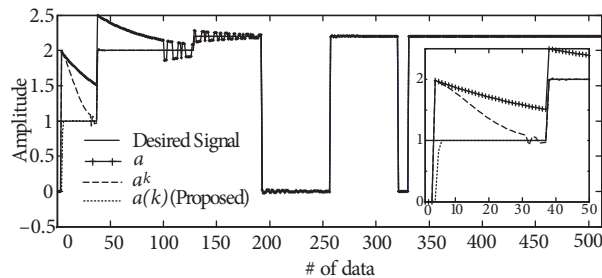


Figure 4. Lyapunov filter outputs for artificial signal ($a = 1.01$ and $a(0) = 1.01$).

The convergence dynamics of the proposed algorithm for the artificial signal are further investigated in terms of mean square error (MSE). As seen from Figure 5, the convergence rate of the proposed algorithm is superior to those of the other two Lyapunov filters. It is found that the proposed algorithm with adjustable adaptation gain rate $a(k)$ is much faster in convergence than the other two Lyapunov filters having constant adaptation gain rate parameters a and a^k , respectively.

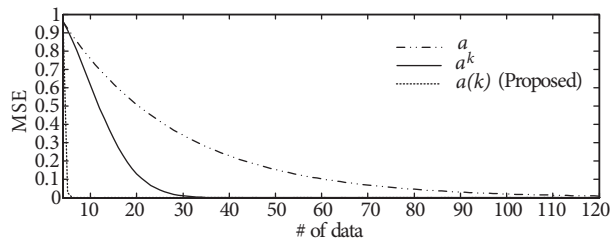


Figure 5. Tracking error comparison among three Lyapunov filters for artificial signal ($a = 1.01$ and $a(0) = 1.01$).

For further comparison, we have also employed a well-known FIR adaptive filter trained with the least mean square (LMS) [3], normalized least mean square (NLMS) [3], and RLS [3] methods for the same artificial signal used in the above simulation example. Their tracking performances are shown in Figures 6a–6c. The best filter parameters are determined after several numbers of trials. As seen from Figures 6a–6c, they could not follow the desired reference signal at all. They have overshoots and fluctuations throughout the reference signal. Although RLS is faster than LMS and NLMS, the output of the RLS adaptive filter diverges from the desired reference signal, and filter parameters are also divergent (Figure 6c).

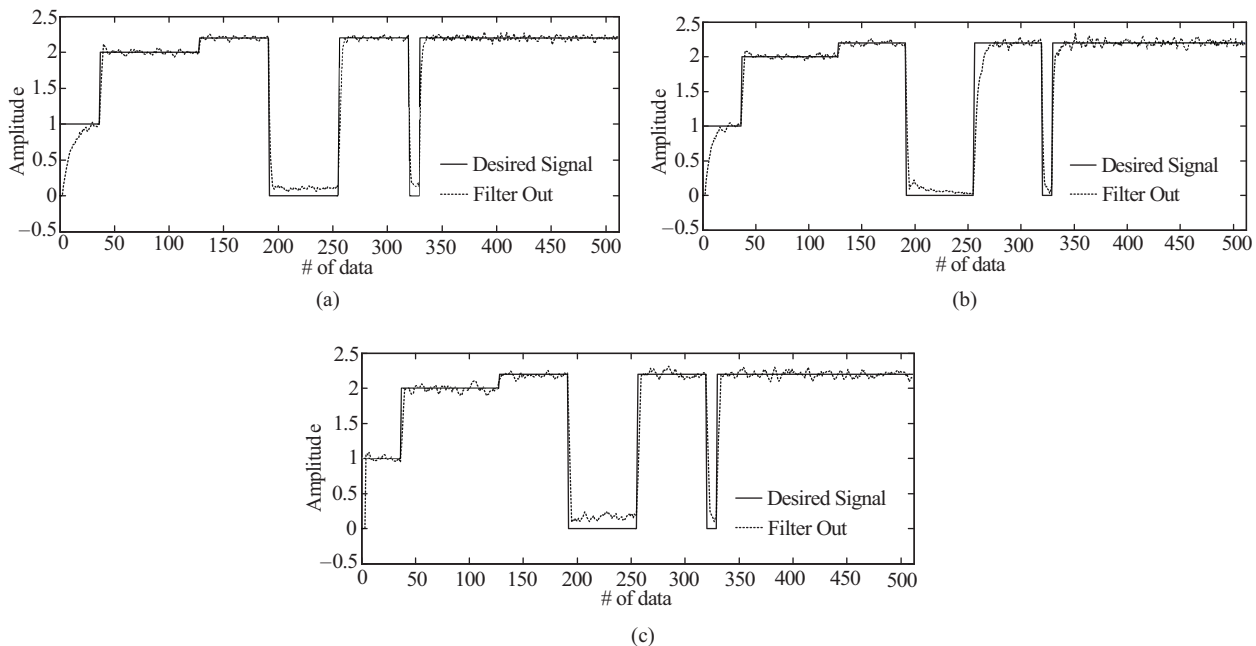


Figure 6. Filter outputs of classical FIR adaptive filter trained with a) LMS ($\mu = 0.02$), b) NLMS ($\mu = 0.2$, $\alpha = 1.3$), and c) RLS ($\lambda = 0.98$, $\sigma = 0.01$).

The performances of the filters are also evaluated by measuring the MSE (dB) values versus different signal-to-noise ratio (SNR) levels. The SNR is varied from -25 dB to 0 dB. As seen from the Figure 7, the proposed algorithm with adjustable adaptation gain rate $a(k)$ (initially $a = 1.01$) shows an excellent performance with about -23 dB (MSE). As a result, the proposed Lyapunov filter is much better than the other filters at all SNR levels. Moreover, the performance of the Lyapunov filters does not change with the noise level increase, indicating a robust behavior.

3.2. Chaotic signals

Desired reference signal $d(k)$ and input signal $\mathbf{x}(k)$ are shown in Figures 8a and 8b. Here, input signal $\mathbf{x}(k) = d(k) + n(k)$ is created at -20 dB random white noise level. A Rossler chaotic time series [14] is used as the desired signal in this simulation example. The filter coefficients $\mathbf{w}_i(k)$ are initialized randomly ($\mathbf{w}_i(0) = 0.01 * \text{randn}(1)$). $a(0)$ is also initialized as $a(0) = 1.01$.

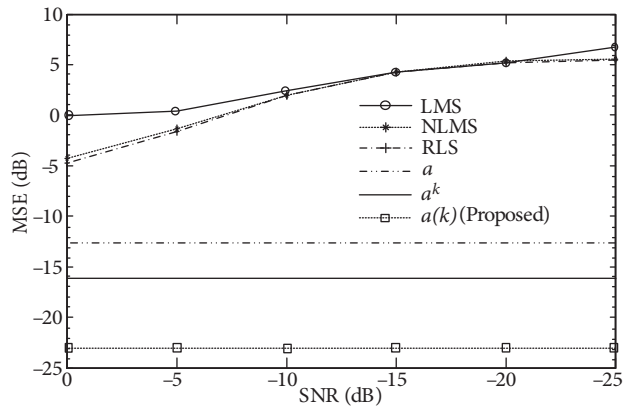


Figure 7. Comparison of the proposed Lyapunov filter with the other filters in terms of MSE (dB) for artificial signal at different noise levels.

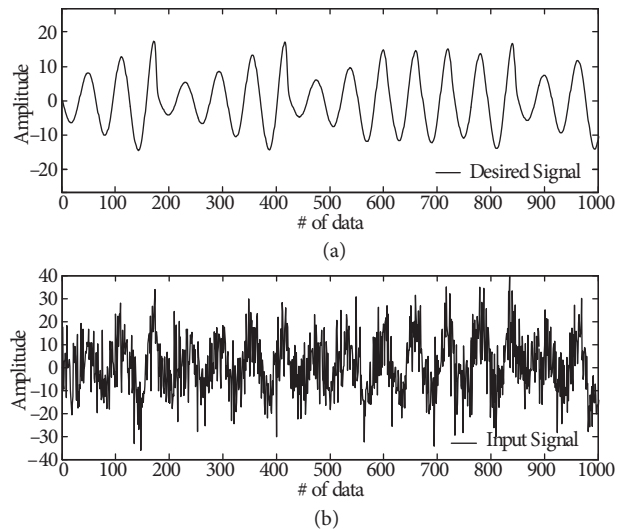


Figure 8. a) Rossler chaotic time series signal (desired reference signal) $d(k)$ and b) noisy input signal $\mathbf{x}(k) = d(k) + n(k)$.

Figure 9 shows the output tracking of the Lyapunov algorithms for $a = 1.01$ and $a(0) = 1.01$. Although the Lyapunov filter with constant adaptation gain rates a and a^k can track the desired reference signal, their outputs are not very good. The overshoots and chattering are easily seen in the filter outputs. On the other hand, the proposed Lyapunov algorithm with its adjustable adaptation gain rate $a(k)$ (initially $a(0) = 1.01$) shows an excellent performance. It has begun to track the desired reference signal a few data points later and then it does not release it.

The convergence dynamics of the proposed algorithm for the Rossler signal are further investigated in terms of MSE. As seen from Figure 10, the convergence rate of the proposed algorithm is superior to those of the other two Lyapunov filters. The proposed Lyapunov algorithm with $a(k)$ is much faster in convergence than the other two approaches.

The performances of the filters for the chaotic Rossler time series are measured in terms of MSE (dB) versus different SNR levels. As seen from Figure 11, the proposed algorithm with adaptation gain rate $a(k)$ (initially $a(0) = 1.01$) shows an excellent performance at around -34 dB (MSE). It is clear that the proposed filter is much better than the other filters at all SNR levels.

It must be mentioned here that two other benchmark chaotic signals, Lorenz and Henon [14], were also simulated in the same way and approximately the same results were observed. Simulation results show that the adjustment capability of the proposed algorithm for adaptation gain rate is the most attractive feature of this study. As a result, the error asymptotically converges to zero, satisfying stability in the sense of Lyapunov, and it could be used as a robust tool in all related engineering applications.

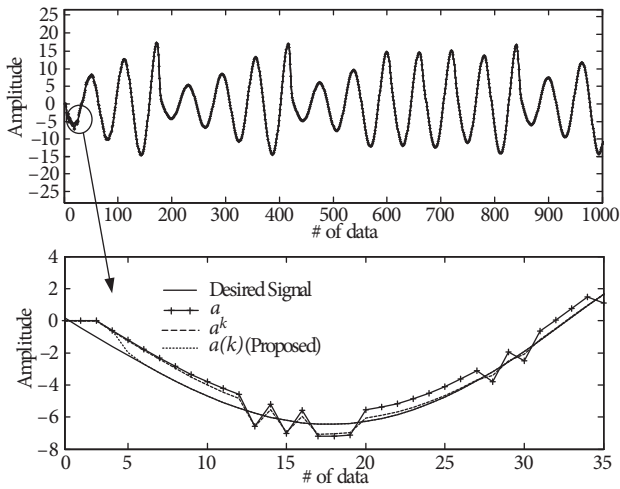


Figure 9. Lyapunov filter outputs for Rossler chaotic time series signal ($a = 1.01$ and $a(0) = 1.01$).

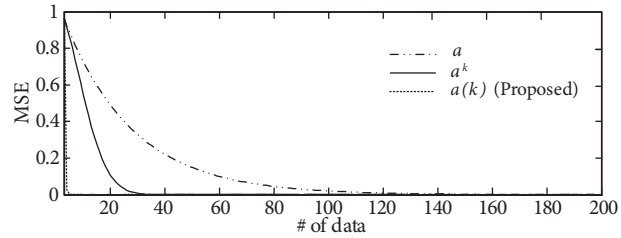


Figure 10. Tracking error comparison among three Lyapunov filters for Rossler signal ($a = 1.01$ and $a(0) = 1.01$).

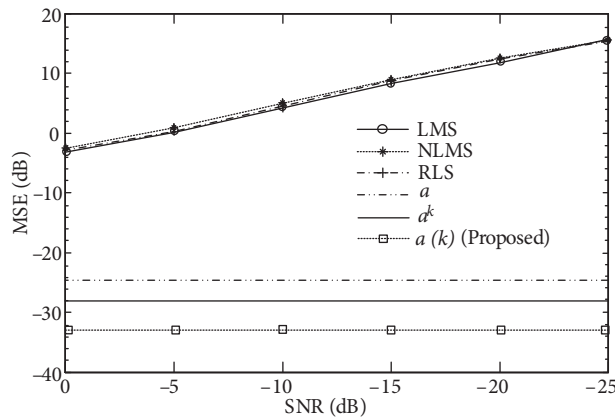


Figure 11. Comparison of the proposed Lyapunov filter with the other filters in terms of MSE (dB) for chaotic Rossler signal at different noise levels.

4. Conclusion

This study presents a novel robust adaptive filter designed in an optimization framework. A new adaptation gain rate, which is a key parameter in convergence dynamics, is obtained. The tracking capability of Lyapunov filtering is improved by using an obtained new adaptation gain rate, $a(k)$, which may adaptively adjust itself depending on sequential tracking square error rate. This feature, contributing to a fast error convergence and a strong robustness with respect to input disturbances, has made the proposed filter algorithm superior to the other Lyapunov and LMS- and RLS-type adaptive filters. The excellent performance of the proposed algorithm is validated by the simulation results.

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