axioms

Article

On Almost Projective Modules

Nil Orhan Ertaş

Department of Mathematics, Bursa Technical University, Bursa 16330, Turkey; orhannil@yahoo.com

Abstract: In this note, we investigate the relationship between almost projective modules and generalized projective modules. These concepts are useful for the study on the finite direct sum of lifting modules. It is proved that; if M is generalized N-projective for any modules M and N, then M is almost N-projective. We also show that if M is almost N-projective and N is lifting, then M is im-small N-projective. We also discuss the question of when the finite direct sum of lifting modules is again lifting.

Keywords: generalized projective module; almost projective modules

MSC: 16D40; 16D80; 13A15

1. Preliminaries and Introduction

Relative projectivity, injectivity, and other related concepts have been studied extensively in recent years by many authors, especially by Harada and his collaborators. These concepts are important and related to some special rings such as Harada rings, Nakayama rings, quasi-Frobenius rings, and serial rings.

Throughout this paper, R is a ring with identity and all modules considered are unitary right R-modules.

Lifting modules were first introduced and studied by Takeuchi [1]. Let M be a module. M is called a lifting module if, for every submodule N of M, there exists a direct summand K of M such that $N/K \ll M/K$. The lifting modules play an important role in the theory of (semi)perfect rings and modules with projective covers. The lifting module is not a generalization of projective modules. In fact, projective modules need not be lifting modules $\mathbb{Z}_{\mathbb{Z}}$. In general, direct sums of lifting modules are not lifting. $\mathbb{Z}/8\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ are lifting $\mathbb{Z}$-modules but $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is not lifting. This fact provides the motivation of this article.

Harada and Tozaki defined the concept of an almost projective module. Then they defined almost injective modules as a dual of almost projective modules. They gave a characterization of Nakayama rings in [2] by using almost projectivity. Let M_1 and M_2 be two modules. M_1 is called almost M_2-projective, if for every epimorphism $f : M_2 \to X$ and every homomorphism $g : M_1 \to X$, either there exists $h : M_1 \to M_2$ with $fh = g$ or there exists a nonzero direct summand N of M_2 and a homomorphism $\gamma : N \to M_1$ with $g\gamma = f\mid_N$. If M_1 is almost M_2-projective for all finitely generated R-modules M_2, then M_1 is called almost projective. Baba and Harada proved that a module $M = \oplus_{i=1}^{n} M_i$, where each M_i is a hollow LE(local endomorphism) module is lifting if and only if M_i is almost M_j-projective for $i \neq j$ and $i, j \in \{1.2, \ldots, n\}$ in [3].

Let $\{M_i \mid i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_{i \in I} M_i$ is called to be exchangeable if, for any direct summand X of M, there exists $M_i' \leq M_i$ for every $i \in I$ such that $M = X \oplus (\oplus M_i')$. A module M is called have (finite) internal exchange property if, any (finite) direct sum decomposition $M = \oplus_{i \in I} M_i$ is exchangeable.

In [4], Mohamed and Müller defined generalized projectivity (dual of the concept of generalized injectivity) as follows. Let A and B be two modules. A is called generalized B-projective if, for any homomorphism $f : A \to X$ and any epimorphism $g : B \to X$, there

check for updates

Citation: Ertaş, N.O. On Almost Projective Modules. *Axioms* **2021**, *10*, 21. https://doi.org/10.3390/axioms10010021

Received: 18 November 2020
Accepted: 5 February 2021
Published: 16 February 2021

exist decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, a homomorphism $h_1 : A_1 \to B_1$ and an epimorphism $h_2 : B_2 \to A_2$ such that $gh_1 = f \mid_{A_1}$ and $fh_2 = g \mid_{B_2}$. The generalized projectivity has roots in the study of direct sums of lifting modules. Kuratomi gave equivalent conditions for a module with exchange decomposition $M = \oplus_{i=1}^{n} M_i$ to be lifting in terms of the relatively generalized projectivity of the direct summand of M in [5]. As a corollary, Kuratomi proved that finite direct sums of lifting modules are again lifting, when the distinct pairs of decomposition are relatively projective.

In [6], Alahmadi and Jain showed that generalized injectivity implies almost injectivity. In this paper, we showed that generalized projectivity implies almost projectivity.

Result 1: Let M and N be right R-modules. If M is generalized N-projective, then M is almost N-projective.

Let M be any module. Consider the following conditions:

(D_2) If $A \leq M$ such that M/A is isomorphic to a summand of M, then A is a summand of M.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Then the module M is called discrete if it is lifting and satisfies the condition (D_2) and it is called quasi-discrete if it is lifting and satisfies the condition (D_3). Since (D_2) implies (D_3), every discrete module is quasi-discrete. In this paper, we give the relation between almost projective modules and some kind of generalized projective modules. We apply these results to a question when the finite direct sum of lifting module is lifting.

Result 2: Let M be a quasi-discrete module and N be a lifting module. If M is almost N-projective, M and N satisfy the descending chain conditions on direct summand, then M is strongly generalized epi-N-projective module.

Result 3: Let $M = M_1 \oplus M_2$ be a module with finite internal exchange property. Assume that for any submodule A of M, if $M = A + M_2$, then $M \neq A + M_1$.Then the following are equivalent:

(1) M is lifting.
(2) M_1 and M_2 are lifting and for every decomposition $M = M_i \oplus M_j$, M_i is generalized M_j-projective for $i \neq j$.
(3) M_1 and M_2 are lifting and for every decomposition $M = M_i \oplus M_j$, M_i is almost M_j-projective for $i \neq j$.
(4) M_1 and M_2 are lifting and for every decomposition $M = M_i \oplus M_j$, M_i is generalized small M_j-projective for $i \neq j$.

Result 4: Let $M_1, \ldots, M_n$ be quasi-discrete and put $M = M_1 \oplus \cdots \oplus M_n$. Then the followings are equivalent.

(1) M is lifting with the (finite) internal exchange property,
(2) M is lifting and the decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable,
(3) M_i is generalized M_j-projective for any $i \neq j \in \{1, \ldots, n\}$.
(4) $M_i \oplus M_j$ is lifting with the finite internal exchange property for $i \neq j \in \{1, \ldots, n\}$,
(5) M_i is strongly generalized epi-M_j-projective and im-small M_j-projective for any $i \neq j \in \{1, \ldots, n\}$,
(6) M_i is generalized epi-M_j-projective and im-small M_j-projective for any $i \neq j \in \{1, \ldots, n\}$,
(7) M_i is strongly generalized epi-M_j-projective and almost M_j-projective for any $i \neq j \in \{1, \ldots, n\}$,
(8) M_i is strongly generalized epi-M_j-projective and generalized small M_j-projective for any $i \neq j \in \{1, \ldots, n\}$,
(9) M_i is generalized epi-M_j-projective and almost M_j-projective for any $i \neq j \in \{1, \ldots, n\}$.

2. Almost Projectivity

In this section, we give the relation between generalized projective modules and almost projective modules.

Theorem 1. *Let M and N be right R-modules. If M is generalized N-projective, then M is almost N-projective.*

Proof. Let $f : M \to X$ be any homomorphism and $g : N \to X$ be any epimorphism for any module X. By assumption there exist decompositions $N = N_1 \oplus N_2$, $M = M_1 \oplus M_2$, a homomorphism $h_1 : M_1 \to N_1$, and an epimorphism $h_2 : N_2 \to M_2$ such that $fh_2 = g\mid_{N_2}$ and $gh_1 = f\mid_{M_1}$. If f can not be lifted to N, then $N \neq N_1$. This means that $N_2 \neq 0$. Define $h : N_2 \to M$ with $h(n_2) = i_2h_2(n_2)$, where $i_2 : M_2 \to M$ is an inclusion map for $n_2 \in N_2$. Now we will show that $fh = g\mid_{N_2}$. Take $n_2 \in N_2$. $fh(n_2) = f(i_2(h_2(n_2))) = f(h_2(n_2)) = g(n_2)$. Hence M is almost N-projective. □

Proposition 1. *Let $H_1, H_2, \ldots H_n$ be hollow modules and $M = H_1 \oplus \cdots \oplus H_n$ exchangeable and N be a quasi-discrete module. If M is almost N-projective then M is generalized N-projective.*

Proof. By the definition of almost projectivity, H_i is also almost N-projective for all $i = 1, 2, \ldots, n$. Clearly H_i are generalized N-projective for all $i = 1, 2, \ldots, n$. By [7] (Proposition 3.2), M is generalized N-projective. □

Now we will give the definitions of generalized epi projective modules and strongly generalized epi projective modules. Generalized epi projective modules were first defined in [8] under the name pseudo cojective modules and the authors gave the characterization of this module. Strongly generalized epi projective modules were first defined in [9].

Definition 1. *M_1 is (strongly) generalized epi-M_2-projective if, for any epimorphism $\varphi : M_1 \to X$ and any epimorphism $\pi : M_2 \to X$, there exist decompositions $M = M_1' \oplus M_1''$, $M_2 = M_2' \oplus M_2''$, a homomorphism (an epimorphism) $\varphi_1 : M_1' \to M_2'$ and an epimorphism $\varphi_2 : M_2'' \to M_1''$ such that $\pi\varphi_1 = \varphi\mid_{M_1'}$ and $\varphi\varphi_2 = \pi\mid_{M_2''}$.*

Clearly, if M_1 is strongly generalized epi-M_2-projective, then M_1 is generalized epi-M_2-projective for modules M_1 and M_2. To give the relation between almost projectivity and strongly generalized epi-projectivity of modules, we need to give some definitions. Let M be a module and let N and K be submodules of M with $N \subseteq K$. N is called a co-essential submodule of K in M if $K/N \ll M/N$ and it is denoted by $N \subseteq_{ce} K$ in M. Let X be a submodule of M. A is called a co-closed submodule in M if A does not have a proper co-essential submodule in M.

Theorem 2. *Let M be a quasi-discrete module and N be a lifting module. If M is almost N-projective, M and N satisfy the descending chain conditions on direct summand, then M is strongly generalized epi-N-projective module.*

Proof. Let $f : M \to X$ and $g : N \to X$ be epimorphisms. Since M and N are lifting, there exist decompositions $M = M_1' \oplus M_1''$ and $N = N_1' \oplus N_1''$ such that $Kerf/M_1' \ll M/M_1'$ and $Kerg/N_1' \ll N/N_1'$. So we see that

$$f(M) = f(M_1') + f(M_1'') = f(M_1''), \qquad g(N) = g(N_1') + g(N_2'') = g(N_2'')$$

and

$$Ker(f\mid_{M_1''}) = Kerf \cap M_1'' \ll M_1'', \qquad Ker(g\mid_{N_1''}) = Kerg \cap N_1'' \ll N_1''.$$

Thus we may assume that, $Kerf \ll M$ and $Kerg \ll N$. Since M is almost N-projective, then either there exists a homomorphism $h : M \to N$ such that $gh = f$ or there exists a decomposition of $N = N_1 \oplus N_2$ and homomorphism $h_2 : N_2 \to M$ such that $fh_2 = g\mid_{N_2}$. Consider the second case.

Since N_2 is lifting and M is amply supplemented, there exists a decomposition $N_2 = N_2' \oplus N_2''$ such that $h_2(N_2')$ is coclosed in M and $h_2(N_2'') \ll M$ by [9] (Lemma 1.6). Since M is lifting, $h_2(N_2')$ is a direct summand of M. Say $h_2(N_2') = M_2$. We also have

$$g(N) = g(N_1) + g(N_2) = g(N_1 \oplus N_2') + g(N_2'') = g(N_1 \oplus N_2') + fh_2(N_2'') = X.$$

Since $h_2(N_2'') \ll M$, $f(h_2(N_2'')) \ll X$ and hence we have $g(N) = g(N_1 \oplus N_2')$. Since $Kerg \ll N$, we have $N = N_1 \oplus N_2'$. This implies that $N_2'' = 0$.

Since M is lifting, there exists a decomposition $M = K \oplus K'$ such that $f^{-1}(g(N_1))/K \ll M/K$. Since N_1 is coclosed in N, then $g(N_1)$ is coclosed in X. Since $K \leq_{ce} f^{-1}(g(N_1)) \leq M$ then $f(K) \leq_{ce} g(N_1) \leq X$. This implies that $f(K) = g(N_1)$. We also have

$$f(f^{-1}(g(N_1)) + M_2) = ff^{-1}(g(N_1)) + f(M_2) = g(N_1) + g(N_2') = g(N) = X = f(M).$$

Since $Kerf \ll M$, $f^{-1}(g(N_1)) + M_2 = M$. Then clearly $K + M_2 = M$. Now we will show that $K \cap M_2 = 0$. By [7] (Lemma 1.7), $g(N_1) \cap g(N_2) \ll X$.

$$K \cap M_2 \leq f^{-1}(g(N_1)) \cap M_2 \leq f^{-1}(g(N_1) \cap g(N_2')) = f^{-1}(g(N_1)) \cap f^{-1}(g(N_2')) \ll M.$$

Hence $K \cap M_2 \ll M$. Since M is quasi-discrete, $K \cap M_2 = 0$.

Now we are in a position there exist decompositions $M = K \oplus M_2$, $N = N_1 \oplus N_2$ and an epimorphism $h_2 : N_2 \to M_2$ with $fh_2 = g\mid_{N_2}$ and $f(K) = g(N_1)$. By [2] (Proposition 4), K is almost N_1-projective, either there exists a decomposition of $N_1 = T_1' \oplus T_1''$ and homomorphism $h_2' : T_1' \to K$ such that $fh_2' = g\mid_{T_1'}$ or there exists a homomorphism $h_1 : K \to N_1$ such that $g\mid_{N_1} h_1 = f$. If the first case hold, by the same manner of the above proof, we get h_2' is an epimorphism. If the second case hold, $gh_1(K) = f(K) = g(N_1)$ implies that $N_1 = h_1(K) + Ker(g\mid_{N_1})$. Since N_1 is lifting, we may assume that $Ker(g\mid_{N_1}) \ll N_1$. Then h_1 is an epimorphism. Since M and N satisfy descending chain conditions on direct summand, this process will stop. Hence we get M is strongly generalized epi-N-projective. □

Hence we can give an immediate result of Theorems 1 and 2.

Corollary 1. *Let M be a quasi-discrete module and N be a lifting module. If M is generalized N-projective, M and N satisfy the descending chain conditions on direct summand, then M is strongly generalized epi-N-projective module.*

3. Generalized Small Projective Modules

In this section, we give the relation between generalized small projective modules and generalized projective modules. Generalized small projective modules were first defined in [8] as follows and the authors gave a characterization of this module.

Definition 2. *M_1 is generalized small M_2-projective if, for any homomorphism $\varphi : M_1 \to X$ with $Im\varphi \ll X$ and any epimorphism $\pi : M_2 \to X$, there exist decompositions $M = M_1' \oplus M_1''$, $M_2 = M_2' \oplus M_2''$, a homomorphism $\varphi_1 : M_1' \to M_2'$ and an epimorphism $\varphi_2 : M_2'' \to M_1''$ such that $\pi\varphi_1 = \varphi\mid_{M_1'}$ and $\varphi\varphi_2 = \pi\mid_{M_2''}$.*

Now we will give the characterization of the generalized small projective module as follows:

Theorem 3 ([8] Proposition 3.3)**.** *Let M_1 and M_2 be R-modules and $M = M_1 \oplus M_2$. Then the following are equivalent:*

(1) M_1 is generalized small M_2-projective.

(2) For every submodule A of M with $(A + M_1)/A \ll M/A$, there exists a decomposition $M = A' \oplus M_1'' \oplus M_2' = A' + M_2$ such that $M_1'' \leq M_1$, $M_2' \leq M_2$.

In general, generalized small projectivity does not imply generalized projectivity.

Example 1 ([10] Example 2.7). *Let S and S' be simple modules with $S \not\cong S'$ and let M and K_1 be uniserial modules such that $M \supset S \supset 0$, $K_1 \supset K_2 \supset S \supset 0$, $M/S \cong S$, $K_1/K_2 \cong S$ and $K_2/S \cong S'$. Then K_1 and M are lifting and K_1 is im-small M-projective. Hence K_1 is generalized small M-projective. But K_1 is not generalized M-projective.*

Proposition 2. *Let K and L be any right R-modules. If K is generalized small-L-projective, then K is generalized small-L^*-projective for any direct summand L^* of L.*

Proof. Define $N = K \oplus L^*$. Let A be a submodule of N such that $(A + K)/A \ll N/A$. This implies that $(A + K)/A \ll M/A$. Since K is generalized small L-projective, there exists a decomposition $M = A' \oplus K' \oplus L' = A' + L$ such that $A' \subseteq A$, $K' \subseteq K$ and $L' \subseteq L$. $N \cap M = N = N \cap (A' \oplus K' \oplus L') = N \cap (A' + L)$. Then we get $N = A' \oplus K' \oplus (N \cap L') = A' + N \cap L$. Since $N = K \oplus L^*$, $N \cap L = (K \oplus L^*) \cap L = L^* \oplus (K \cap L) = L^*$. Then $N \cap L' \subseteq N \cap L = L^*$. Clearly $A' + N \cap L = A' + L^*$. Then K is generalized small-L^*-projective. □

Proposition 3. *Let M be a lifting module with finite internal exchange property. Then for every decomposition $M = M_1 \oplus M_2$, M_i is generalized small M_j-projective for $i \neq j$ and $i, j \in \{1, 2\}$.*

Proof. It is obtained from [4] (Proposition 3.5). □

Proposition 4. *Let M be a quasi-discrete module. Then for every decomposition $M = M_1 \oplus M_2$, M_i is generalized small M_j-projective for $i \neq j$ and $i, j \in \{1, 2\}$.*

Proof. It is obtained by [11] (Proposition 4.23). □

Definition 3. *Let M and N be right R-modules. M is called im-small N-projective if for any submodule A of N, any homomorphism $f : M \to N/A$ with $Imf \ll N/A$ can be lifted to a homomorphism $g : M \to N$.*

Now we give the relation between generalized small modules and im-small modules which is in [12] (Lemma 2.10). For the sake of completeness, we will give the proof of this lemma.

Lemma 1. *Let M_1 be any module and M_2 be a lifting module. If M_1 is generalized small M_2-projective, then M_1 is im-small M_2-projective.*

Proof. Let $\pi : M_2 \to X$ be an epimorphism and $\phi : M_1 \to X$ be a homomorphism with $Im\phi \ll X$. Since M_2 is lifting, there exists a decomposition $M_2 = A \oplus B$ such that $Ker\pi/B \ll M_2/B$. Then we have $\pi(M_2) = \pi(A) + \pi(B) = \pi(A)$. And we also have $Ker\pi \mid_A = \ker \pi \cap A \ll A$. Hence we may assume that $Ker\pi \ll M_2$ by [5] (Proposition 2.1). Since $\pi : M_2 \to X$ is a small epimorphism, for any submodule C of M_2, $\pi(C) \ll X$ if and only if $C \ll M_2$. Hence we cannot have a map from a direct summand of M_2 to M_1 satisfying the condition for M_1 to be generalized M_2-projective. Hence M_1 is im-small M_2-projective. □

Theorem 4. *Let M and N be any right R-modules. If M is an almost N-projective module and N is lifting, then M is im-small N-projective.*

Proof. Let $g : N \to X$ be an epimorphism and let $f : M \to X$ be a homomorphism with $Imf \ll X$. Since N is lifting, we may assume that $Kerg \ll N$ as in the proof of Theorem 2. Since M is almost N-projective, there exists a homomorphism $h : M \to N$ such that $gh = f$ or there exists a decomposition of $N = N_1 \oplus N_2$ and homomorphism $h_2 : N_2 \to M$ such that $fh_2 = g\mid_{N_2}$. Consider the second case. Since $Imf \ll X$, $g(N_2) = fh(N_2) \ll X$. Then $g(N) = g(N_1) + g(N_2) = X$ implies that $g(N) = g(N_1)$. Since $Kerg \ll N$, $N = N_1$. Hence $N_2 = 0$. Therefore we have the first case. This completes the proof. □

Now we can give an immediate result of the Theorem 4, Theorem 1 and Lemma 1 as a generalization of [9] (Proposition 2.7).

Corollary 2. *Let M and N be lifting modules with the finite internal exchange property. Then M is generalized N-projective if and only if M is strongly generalized epi-N-projective and generalized small N-projective if and only if M is strongly generalized epi-N-projective and almost N-projective.*

Lemma 2 ([8] Lemma 4.9)**.** *Let M_1 and M_2 be modules and $M = M_1 \oplus M_2$. Assume that for any submodule A of M if $M = A + M_2$, then $M \neq A + M_1$. If M_1 is generalized small M_2-projective, then M_1 is generalized M_2-projective.*

Now we can apply this result when a finite direct sum of lifting modules is lifting.

Theorem 5. *Let $M = M_1 \oplus M_2$ be a module with finite internal exchange property. Assume that for any submodule A of M, if $M = A + M_2$, then $M \neq A + M_1$.Then the following are equivalent:*

(1) M is lifting.
(2) M_1 and M_2 are lifting and for every decomposition $M = M_i \oplus M_j$, M_i is generalized M_j-projective for $i \neq j$.
(3) M_1 and M_2 are lifting and for every decomposition $M = M_i \oplus M_j$, M_i is almost M_j-projective for $i \neq j$.
(4) M_1 and M_2 are lifting and for every decomposition $M = M_i \oplus M_j$, M_i is generalized small M_j-projective for $i \neq j$.

Proof. (1) ⇔ (2) ⇔ (4) They are clear [8] (Lemma 4.9).
(2) ⇒ (3) It is clear by Theorem 1.
(3) ⇒ (4) It is clear by definition and Theorem 4. □

Now we can give a result which is a generalization of [9] (Theorem 2.9).

Theorem 6. *Let M_1 and M_2 be lifting modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then the following are equivalent:*

(1) M is lifting with the finite exchange property.
(2) M is lifting and the decomposition $M = M_1 \oplus M_2$ is exchangeable.
(3) M_1 is generalized M_2-projective and M_2 is im-small M_1-projective.
(4) M_2 is generalized M_1-projective and M_1 is im-small M_2-projective.
(5) (M_i) is strongly generalized epi-M_j-projective and im-small M_j-projective for $i \neq j$.
(6) (M_i) is strongly generalized epi-M_j-projective and almost M_j-projective for $i \neq j$.
(7) (M_i) is strongly generalized epi-M_j-projective and generalized small M_j-projective for $i \neq j$.

Proof.
(1) ⇔ (2) By [5] (Theorem 3.7).
(2) ⇔ (3) ⇔ (4) ⇔ (5) By [9] (Theorem2.9).
(5 ⇔ (6) ⇔ (7) By Corollary 2. □

Theorem 7. *Let $M_1, \ldots, M_n$ be quasi-discrete and put $M = M_1 \oplus \cdots \oplus M_n$. Then the following are equivalent:*

(1) M is lifting with the (finite) internal exchange property,
(2) M is lifting and the decomposition $M = M_1 \oplus \cdots \oplus M_n$ is exchangeable,
(3) M_i is generalized M_j-projective for any $i \neq j \in \{1, \ldots, n\}$.
(4) $M_i \oplus M_j$ is lifting with the finite internal exchange property for $i \neq j \in \{1, \ldots, n\}$,

(5) M_i *is strongly generalized epi-*M_j*-projective and im-small* M_j*-projective for any* $i \neq j \in \{1, \ldots, n\}$,
(6) M_i *is generalized epi-*M_j*-projective and im-small* M_j*-projective for any* $i \neq j \in \{1, \ldots, n\}$,
(7) M_i *is strongly generalized epi-*M_j*-projective and almost* M_j*-projective for any* $i \neq j \in \{1, \ldots, n\}$,
(8) M_i *is strongly generalized epi-*M_j*-projective and generalized small* M_j*-projective for any* $i \neq j \in \{1, \ldots, n\}$,
(9) M_i *is generalized epi-*M_j*-projective and almost* M_j*-projective for any* $i \neq j \in \{1, \ldots, n\}$.

Proof.
(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows by [9] (Theorem 2.16).
(3) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) It is clear by Corollary 2.
(3) $\Rightarrow$ (9) It is clear by definition and Theorem 1.
(9) $\Rightarrow$ (6) It is clear by Theorem 4. □

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to express their gratefulness to the referees' suggestions which improved the presentation of the paper.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Takeuchi, T. On cofinite-dimensional modules. *Hokkaido Math. J.* **1976**, *5*, 1–43. [CrossRef]
2. Harada, M.; Tozaki, A. Almost *M*-projectives and Nakayama rings. *J. Algebra* **1989**, *122*, 447–474. [CrossRef]
3. Baba, Y.; Harada, M. On almost *M*-projectives and almost *M*-injectives. *Tsukuba J. Math.* **1990**, *14*, 53–69. [CrossRef]
4. Mohammed, S.H.; Müller, B.J. Cojective modules. *J. Egypt. Math. Sci.* **2004**, *12*, 83–96.
5. Kuratomi, Y. On direct sums of lifting modules and internal exchage property. *Commun. Algebra* **2005**, *3*, 1795–1804. [CrossRef]
6. Alahmadi, A.; Jain, S.K. A note on almost injective modules. *Math. J. Okayama Univ.* **2009**, *51*, 101–109.
7. Kuratomi, Y. Generalized projectivity of quasi-discrete module. *Int. Electron. J. Algebra* **2008**, *3*, 125–134.
8. Orhan, N.; Keskin Tütüncü, D. Characterization of lifting modules in terms of cojective modules and the class of $\mathcal{B}(\mathcal{M}, \mathcal{X})$. *Int. J. Math.* **2005**, *16*, 647–660. [CrossRef]
9. Keskin Tütüncü, D.; Kuratomi, Y. On generalized epi-projective modules. *Math. J. Okayama Uni.* **2010**, *52*, 111–122.
10. Keskin Tütüncü, D.; Kuratomi, Y. On Epi-projective Modules. *East West J. Math.* **2008**, *10*, 27–35.
11. Mohammed, S.H.; Müller, B.J. *Continuous and Discrete Module*; London Mathematical Society Lecture Note Series 147; Cambridge University Press: Cambridge, UK. 1990.
12. Orhan Ertaş, N. Relation betweem almost projective modules and projective modules. In Proceedings of the 10th International European Conference on Mathematics, Engineering, Natural and Medical Sciences, Izmir, Turkey, 14–15 December 2019.