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Sum-edge characteristic polynomials of graphs

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ABSTRACT
Modelling a chemical compound by a (molecular) graph helps us to obtain some required information about the chemical and physical properties of the corresponding molecular structure. Linear algebraic notions and methods are used to obtain several properties of graphs usually by the help of some graph matrices and these studies form an important sub area of Graph Theory called spectral graph theory. In this paper, we deal with the sum-edge matrices defined by means of vertex degrees. We calculate the sum-edge characteristic polynomials of several important graph classes by means of the corresponding sum-edge matrices.

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1. Introduction

Let $G = (V, E)$ be a graph with $| V(G) |= n$ vertices and $| E(G) |= m$ edges. For a vertex $v \in V(G)$, we denote the degree of v by d_v or $d_G(v)$. In particular, a vertex with degree one is called a pendant vertex. With slight abuse of language, one can use the term "pendant edge" for an edge having a pendant vertex. If u and v are two vertices of G connected by an edge e, then this situation is denoted by $e = uv$. In such a case, the vertices u and v are called adjacent vertices and the edge e is said to be incident with u and v. The study of adjacency and incidency with the help of corresponding matrices is a well known application of Graph Theory to Molecular Chemistry and the sub area of Graph Theory dealing with the energy of a graph is called Spectral Graph Theory which uses linear algebraic methods to calculate eigenvalues of a graph resulting in the molecular energy of that graph. In that sense, matrices are very helpful in the spectral study of graphs modelling some chemical structures. Apart from three most important kinds of matrices, that are Laplacian, adjacency and incidency matrices, there are nearly one hundred types of graph matrices, some giving important information about the molecules that are modelled by the corresponding graph, see e.g. [1, 2] for general notions on the graph matrices. In [3–5], some formulae and recurrence relations on spectral polynomials of some graphs were calculated.

As mentioned above, one of the ways of studying graphs is to make use of the matrices corresponding the graph and for this aim, a large number of matrices have been defined and used. The most popular ones are the adjacency, incidency and Laplacian matrices. In the recent years, some mew matrices by means of the values of several topological indices have been defined. In this paper, we shall consider the sum-edge characteristic polynomials obtained as the characteristic polynomial of the sum-edge matrix.

2. Sum-Edge matrices and characteristic polynomials

One of these matrices called the sum-edge matrix of G is a square $n \times n$ matrix $SM^e(G) = [a_{ij}]_{n \times n}$ determined by the adjacency of vertices as follows:

$$a_{ij} = \begin{cases} d_G(v_i) + d_G(v_j), & \text{if the vertices } v_i \text{ and } v_j \\ & \quad \text{are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

For a graph G, we shall denote the sum-edge characteristic polynomial obtained by taking the sum-edge matrix of G instead of the classical adjacency matrix by $P_G^{se}(\lambda)$. In this paper, we shall determine this sum-edge characteristic polynomial of some well-known graph classes and also give some general results for r-regular graphs.

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There is a special matrix which appears in many calculations in this paper. This matrix is

$$\Delta_n = \begin{bmatrix} 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 4 & 0 & 4 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 4 & 0 \\ 0 & 0 & 0 & \cdots & 4 & 0 & 4 \\ 0 & 0 & 0 & \cdots & 0 & 4 & 0 \end{bmatrix}.$$

The characteristic polynomial corresponding to this matrix is denoted by $P_{\Delta_n}(\lambda)$ and is equal to

$$P_{\Delta_n}(\lambda) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{4k} \binom{n-k}{n-2k} \lambda^{n-2k}.$$

We now recall a well-known property of determinants:

Lemma 2.1: *If we divide a matrix into four block matrices*

$$\left[\begin{array}{c|c} A & 0 \\ \hline C & B \end{array}\right],$$

or

$$\left[\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right]$$

where A and B are square matrices, the determinant of this matrix is equal to $|A|\,|B|$.

We shall now determine sum-edge characteristic polynomial of an r-regular graph. For $r > 2$, because of the variety of r-regular graphs, it seems that there is no unique method to calculate $P_{\Delta_n}(\lambda)$. Only when $r = 2$, we can calculate this sum-edge characteristic polynomial easily. As all 2-regular graphs are either cyclic graphs when the graph is connected, or each component is a cycle when the graph is not connected. In the latter case, we can easily obtain this sum-edge characteristic polynomial using determinant properties if we know this polynomial for a connected 2-regular graph:

Theorem 2.1: *The formula for the sum-edge characteristic polynomial of the cycle graph C_n obtained by means of the sum-edge matrix is*

$$P^{se}_{C_n}(\lambda) = -2^{2n+1} - \sum_{k=0}^{\lfloor (n-2)/2 \rfloor} (-1)^k 2^{4k+5} \times \binom{n-2-k}{n-2-2k} \lambda^{n-2-2k} + \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k 2^{4k} \times \binom{n-1-k}{n-1-2k} \lambda^{n-2k}.$$

Proof: The sum-edge matrix of C_n is

$$SM^e(C_n) = \begin{bmatrix} 0 & 4 & 0 & 0 & 0 & \cdots & 0 & 0 & 4 \\ 4 & 0 & 4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & 4 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & \cdots & 0 & 4 & 0 \end{bmatrix}.$$

Hence

$$|\lambda I - SM^e(C_n)| = \begin{bmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ -4 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{bmatrix}.$$

If we calculate this determinant according to the last row, we get

$$= (-1)^{n+2} \cdot 4 \begin{vmatrix} -4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \\ \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \end{vmatrix}$$

$$+ (-1)^{2n} \cdot 4 \begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & -4 \end{vmatrix}$$

$$+ (-1)^{2n} \cdot \lambda \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

If we calculate the determinant of the first matrix with respect to the first row and calculate the determinant of the second matrix according to the last column, then we get

$$= (-1)^{n+5} \cdot 16 \begin{vmatrix} -4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \end{vmatrix}$$

$$+(-1)^{2n+3}\cdot 16 \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+(-1)^{3n+1}\cdot 16 \begin{vmatrix} -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \end{vmatrix}$$

$$+(-1)^{4n-1}\cdot 16 \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+\lambda \begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

One can find the dimensions of the matrices are $(n-2)\times(n-2)$, $(n-2)\times(n-2)$, $(n-2)\times(n-2)$, $(n-2)\times(n-2)$, and $(n-1)\times(n-1)$, respectively.

Thanks to the determinant property of lower triangular matrices, we get

$$\begin{aligned} P^{se}_{C_n}(\lambda) &= (-1)^{n+5}16(-4)^{n-2} + (-1)^{2n+3}16P_{\Delta_{n-2}}(\lambda) \\ &\quad + (-1)^{3n+1}16(-4)^{n-2} \\ &\quad + (-1)^{4n-1}16P_{\Delta_{n-2}}(\lambda) + \lambda P_{\Delta_{n-1}}(\lambda)\cdot \\ &= -32\cdot 4^{n-2} - 32P_{\Delta_{n-2}}(\lambda) + \lambda P_{\Delta_{n-1}}(\lambda). \end{aligned}$$

Hence we get

$$\begin{aligned} P^{se}_{C_n}(\lambda) &= -2^{2n+1} - \sum_{k=0}^{\lfloor (n-2)/2\rfloor} (-1)^k 2^{4k+5} \\ &\quad \times \binom{n-2-k}{n-2-2k}\lambda^{n-2-2k} \\ &\quad + \sum_{k=0}^{\lfloor (n-1)/2\rfloor} (-1)^k 2^{4k} \\ &\quad \times \binom{n-1-k}{n-1-2k}\lambda^{n-2k}. \end{aligned}$$

■

Now we calculate the sum-edge characteristic polynomial of some other graph classes. We have already considered the cycle graphs during our general study of regular graphs. Now we start with the path graph P_n.

Theorem 2.2: *The formula for the sum-edge characteristic polynomial of the path graph P_n is*

$$\begin{aligned} P^{se}_{P_n}(\lambda) &= \sum_{k=0}^{\lfloor (n-2)/2\rfloor} (-1)^k 2^{4k}\binom{n-2-k}{n-2-2k}\lambda^{n-2k} \\ &\quad - 18\sum_{k=0}^{\lfloor (n-3)/2\rfloor} (-1)^k 2^{4k}\binom{n-3-k}{n-3-2k}\lambda^{n-2-2k} \\ &\quad + 81\sum_{k=0}^{\lfloor (n-4)/2\rfloor} (-1)^k 2^{4k}\binom{n-4-k}{n-4-2k}\lambda^{n-4-2k} \end{aligned}$$

if $n \geq 4$ and $P^{se}_{P_1}(\lambda) = \lambda$, $P^{se}_{P_2}(\lambda) = \lambda^2 - 4$, $P^{se}_{P_3}(\lambda) = \lambda^3 - 18\lambda$.

Proof: The sum-edge characteristic polynomial $P^{se}_{P_n}(\lambda)$ of a path graph P_n is obtained by taking the determinant

$$P^{se}_{P_n}(\lambda) = \begin{vmatrix} \lambda & -3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -3 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -3 & \lambda \end{vmatrix}.$$

For $n=1$, the one dimensional matrix whose single entry is 0. Therefore $P^{se}_{P_1}(\lambda) = \lambda$. For $n=2,3$, the proof is obvious.

If we calculate $P^{se}_{P_n}(\lambda)$ for $n \geq 4$ with respect to the first row, we get

$$P^{se}_{P_n}(\lambda) = \lambda \begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -3 & \lambda \end{vmatrix}$$

$$+3 \begin{vmatrix} -3 & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -3 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -3 & \lambda \end{vmatrix}.$$

If we calculate the determinant of the second matrix according to the first column, we get

$$P^{se}_{P_n}(\lambda)=\lambda\begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -3\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -3 & \lambda \end{vmatrix}$$

$$-9\begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -3\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -3 & \lambda \end{vmatrix}.$$

The dimension of the first matrix is $(n-1)\times(n-1)$ and the dimension of the second matrix is $(n-2)\times(n-2)$. Let's calculate the determinant of the first and second matrices with respect to the last row, then we get

$$=-3\lambda(-1)^{2n-3}\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & 0\\ 0 & 0 & 0 & \cdots & 0 & -4 & -3 \end{vmatrix}$$

$$+(-1)^{2n-2}\lambda^2\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+27(-1)^{2n-5}\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & 0\\ 0 & 0 & 0 & \cdots & 0 & -4 & -3 \end{vmatrix}$$

$$+-9\lambda(-1)^{2n-4}$$

$$\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}.$$

If we calculate the determinants of first and third matrices according to the last column, we get

$$=-9\lambda\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+\lambda^2\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+81\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$-9\lambda\begin{vmatrix} \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\ -4 & \lambda & -4 & \cdots & 0 & 0 & 0\\ 0 & -4 & \lambda & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot\\ 0 & 0 & 0 & \cdots & \lambda & -4 & 0\\ 0 & 0 & 0 & \cdots & -4 & \lambda & -4\\ 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}.$$

The dimension of the first and last matrices are $(n-3)\times(n-3)$; the dimension of the second matrix is $(n-2)\times(n-2)$, and finally the dimension of the third matrix is $(n-4)\times(n-4)$.

$$P^{se}_{P_n}(\lambda)=\lambda^2 P_{\Delta_{n-2}}(\lambda)-18\lambda P_{\Delta_{n-3}}(\lambda)+81P_{\Delta_{n-4}}(\lambda).$$

Consequently, we can write the formula for sum-edge characteristic polynomial of the path graph P_n as follows:

$$\begin{aligned} P^{se}_{P_n}(\lambda)=&\sum_{k=0}^{\lfloor (n-2)/2\rfloor}(-1)^k 2^{4k}\binom{n-2-k}{n-2-2k}\lambda^{n-2k}\\ &-18\sum_{k=0}^{\lfloor (n-3)/2\rfloor}(-1)^k 2^{4k}\binom{n-3-k}{n-3-2k}\lambda^{n-2-2k}\\ &+81\sum_{k=0}^{\lfloor (n-4)/2\rfloor}(-1)^k 2^{4k}\binom{n-4-k}{n-4-2k}\lambda^{n-4-2k}. \end{aligned}$$

■

Now we can obtain the sum-edge characteristic polynomial of the tadpole graphs:

Theorem 2.3: *The formula for the sum-edge characteristic polynomial of the tadpole graph $T_{r,s}$ obtained by means of the sum-edge matrix is*

$$P^{se}_{T_{r,s}}(\lambda)=\begin{cases} P_{\Delta_{r-1}}(\lambda)\left(\lambda^2-16\right) \\ \quad -50\lambda P_{\Delta_{r-2}}(\lambda)-25\cdot 2^{2r-3}\lambda, \quad s=1 \\ P_{\Delta_{r-1}}(\lambda)(\lambda^3-34\lambda) \\ \quad -50(\lambda^2-9)(P_{\Delta_{r-2}}(\lambda)+2^{2r-4}), \quad s=2 \\ \lambda P_{\Delta_{s-1}}(\lambda)\left[\lambda P_{\Delta_{r-1}}(\lambda)\right. \\ \quad \left. -\ 50\left(P_{\Delta_{r-2}}(\lambda)+2^{2r-4}\right)\right] \\ \quad +P_{\Delta_{s-2}}(\lambda)\left[-34\lambda P_{\Delta_{r-1}}(\lambda)\right. \\ \quad \left. +\ 450\left(P_{\Delta_{r-2}}(\lambda)+2^{2r-4}\right)\right] \\ \quad +225P_{\Delta_{s-3}}(\lambda)P_{\Delta_{r-1}}(\lambda), \quad s>2. \end{cases}$$

Proof: Let $s=1$. First note that

$$P^{se}_{T_{r,1}}(\lambda)=\begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & \lambda & -5 & 0 & 0 & \cdots & 0 & 0 & -5 \\ 0 & -5 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & -5 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}.$$

If we calculate this determinant with respect to second row, we get

$$=4\begin{vmatrix} -4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -5 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ -5 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+\lambda\begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+5\begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -5 & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & -5 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}$$

$$+(-1)^{r+3}(-5)$$

$$\begin{vmatrix} \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -5 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & -5 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}.$$

If we calculate the third determinant with respect to the second row and calculate the last determinant according to the last row, we get

$$P^{se}_{T_{r,1}}(\lambda)=-16P_{\Delta_{r-1}}(\lambda)+\lambda^2P_{\Delta_{r-1}}(\lambda)$$

$$+5\left(-5\begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -6 \\ 0 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}\right.$$

$$\left.+4\begin{vmatrix} \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \\ 0 & -5 & 0 & 0 & \cdots & 0 & 0 & -4 \end{vmatrix}\right)$$

$$+5\cdot(-1)^{r+4}$$

$$\left(5\cdot(-1)^{r+3}\begin{vmatrix} \lambda & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & \lambda & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \end{vmatrix}\right.$$

$$\left.+-4\begin{vmatrix} \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -5 & \lambda & -4 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -4 & \lambda & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & -4 & \lambda \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 \end{vmatrix}\right).$$

Then we get

$$P^{se}_{T_{r,1}}(\lambda)=P_{\Delta_{r-1}}(\lambda)\left(\lambda^2-16\right)-50\lambda P_{\Delta_{r-2}}(\lambda)$$
$$-25\cdot 2^{2r-3}\lambda.$$

Let $s=2$. The form of $P^{se}_{T_{r,2}}(\lambda)$ is

$$P^{se}_{T_{r,2}}(\lambda)$$

$$=\begin{vmatrix} \lambda & -3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -3 & \lambda & -5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -5 & \lambda & -5 & 0 & 0 & 0 & \cdots & 0 & 0 & -5 \\ 0 & 0 & -5 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \lambda & -4 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4 \\ 0 & 0 & -5 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda \end{vmatrix}.$$

If we calculate this determinant with respect to the third row, then we get

$$
= 5\begin{vmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
-3 & -5 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & -5 & \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\
. & . & . & \cdots & . & . & . & . & . & . & .\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4 & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda & -4\\
0 & -5 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}
$$

$$
+\lambda\begin{vmatrix}
\lambda & -3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
-3 & \lambda & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & \lambda & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & -4 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\
. & . & . & \cdots & . & . & . & . & . & . & .\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4 & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda & -4\\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}
$$

$$
+5\begin{vmatrix}
\lambda & -3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
-3 & \lambda & -5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & -5 & -4 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0\\
. & . & . & . & \cdots & . & . & . & . & . & .\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \lambda & -4 & 0\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -4\\
0 & 0 & -5 & 0 & \cdots & 0 & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}
$$

$$
+(-1)^{r+5}(-5)
$$

$$
\begin{vmatrix}
\lambda & -3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
-3 & \lambda & -5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & -5 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0\\
. & . & . & . & \cdots & . & . & . & . & .\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \lambda & -4\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda\\
0 & 0 & -5 & 0 & \cdots & 0 & 0 & 0 & 0 & -4
\end{vmatrix}
$$

If we calculate the determinant of the third matrix according to the third row, this third determinant becomes

$$
\begin{vmatrix}
\lambda & -3 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
-3 & \lambda & -5 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & -4 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & \lambda & -4 & 0 & 0 & \cdots & 0 & 0\\
0 & 0 & 0 & -4 & \lambda & -4 & 0 & \cdots & 0 & 0\\
. & . & . & . & \cdots & . & . & . & . & .\\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -4 & \lambda & -4\\
0 & 0 & -5 & 0 & \cdots & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}.
$$

If we continue to reduce this determinant according to the third row until the lower right block matrix is in the form $\left[\begin{smallmatrix} 0 & -4\\ -5 & \lambda\end{smallmatrix}\right]$, then the third determinant is calculated. Similarly continuing, we get

$$
\begin{aligned}
P^{se}_{T_{r,2}}(\lambda) = {} & P_{\Delta_{r-1}}(\lambda)(\lambda^3 - 34\lambda)\\
& - 50(\lambda^2 - 9)(P_{\Delta_{r-2}}(\lambda) + 2^{2r-4}).
\end{aligned}
$$

Finally, let $s > 2$. The form of $P^{se}_{T_{r,s}}(\lambda)$ is

$$
P^{se}_{T_{r,s}}(\lambda)
$$

$$
= \begin{vmatrix}
\lambda & -3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-3 & \lambda & -4 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .\\
0 & -4 & \lambda & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .\\
. & \cdots & . & . & . & . & . & . & . & . & . & . & . & .\\
0 & \cdots & 0 & 0 & -4 & \lambda & -4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & \cdots & 0 & 0 & 0 & -4 & \lambda & -5 & 0 & 0 & 0 & \cdots & 0 & 0\\
0 & \cdots & 0 & 0 & 0 & 0 & -5 & \lambda & -5 & 0 & 0 & \cdots & 0 & -5\\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & -5 & \lambda & -4 & 0 & \cdots & 0 & 0\\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -4 & \lambda & -4 & \cdots & 0 & 0\\
. & \cdots & . & . & . & . & . & . & . & . & . & . & . & .\\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & \lambda & -4\\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & -4 & \lambda
\end{vmatrix}.
$$

If we calculate the determinant of this matrix according to the $(s+1)$th row, and continuing as above, we get

$$
\begin{aligned}
P^{se}_{T_{r,s}}(\lambda) = {} & \lambda P_{\Delta_{s-1}}(\lambda)[\lambda P_{\Delta_{r-1}}(\lambda)\\
& - 50\left(P_{\Delta_{r-2}}(\lambda) + 2^{2r-4}\right)]\\
& + P_{\Delta_{s-2}}(\lambda)\left[-34\lambda P_{\Delta_{r-1}}(\lambda)\right.\\
& \left. + 450\left(P_{\Delta_{r-2}}(\lambda) + 2^{2r-4}\right)\right]\\
& + 225 P_{\Delta_{s-3}}(\lambda) P_{\Delta_{r-1}}(\lambda).
\end{aligned}
$$

Hence the result follows. ■

Recall that the determinant of any matrix A is equal to the product of its eigenvalues λ_k. So we have

Theorem 2.4: *The formula for the sum-edge characteristic polynomial of the star graph S_n obtained by means of the sum-edge matrix is*

$$
P^{se}_{S_n}(\lambda) = \lambda^{n-2}\left[\lambda^2 - n^2(n-1)\right].
$$

Proof: Let $n = 2$. Then

$$
|SM^e(S_2)| = -4 \neq 0.
$$

So, none of the eigenvalues can be zero, then the form of $P^{se}_{S_n}(\lambda)$ is

$$
P^{se}_{S_n}(\lambda) = \begin{vmatrix}
\lambda & -n & 0 & 0 & 0 & \cdots & 0 & 0\\
-n & \lambda & -n & -n & -n & \cdots & -n & -n\\
0 & -n & \lambda & 0 & 0 & \cdots & 0 & 0\\
0 & -n & 0 & 0 & 0 & \cdots & 0 & 0\\
. & . & . & \cdots & . & . & 0 & .\\
0 & -n & 0 & \cdots & 0 & 0 & \lambda & 0\\
0 & -n & 0 & \cdots & 0 & 0 & 0 & \lambda
\end{vmatrix}.
$$

If we divide the matrix into block matrices so that the upper left one is 2×2 and the lower right one is $(n -$

2) $\times (n-2)$ and if we use the following elementary row operations

$$\begin{gathered}\frac{n}{\lambda}R_3 + R_2 \longrightarrow R_2\\ \frac{n}{\lambda}R_4 + R_2 \longrightarrow R_2\\ \cdots\\ \frac{n}{\lambda}R_n + R_2 \longrightarrow R_2.\end{gathered}$$

If the process continue, then we get

$$P^{se}_{S_n}(\lambda) = \begin{vmatrix} \lambda & -n & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ -n & \lambda - \dfrac{(n-2)n^2}{\lambda} & 0 & 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & -n & \lambda & 0 & 0 & 0 & \cdots & 0 & 0\\ . & . & . & . & . & . & . & . & .\\ 0 & -n & 0 & 0 & 0 & \cdots & 0 & \lambda & 0\\ 0 & -n & 0 & 0 & 0 & \cdots & 0 & 0 & \lambda \end{vmatrix}.$$

By Lemma 2.1, we obtain

$$\begin{aligned} &= \left[\lambda^2 - (n-2)n^2 - n^2\right]\lambda^{n-2}\\ &= \lambda^{n-2}\left[\lambda^2 - n^2(n-1)\right]. \end{aligned}$$

Let now $n \geq 3$.

$$|\mathrm{SM}^e(S_n)| = \begin{vmatrix} 0 & n & 0 & 0 & 0 & \cdots & 0 & 0\\ n & 0 & n & n & n & \cdots & n & n\\ 0 & n & 0 & 0 & 0 & \cdots & 0 & 0\\ . & . & . & . & . & . & . & .\\ 0 & n & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & n & 0 & \cdots & 0 & 0 & 0 & 0 \end{vmatrix}.$$

Thanks to elementary row operations, we see that

$$|\mathrm{SM}^e(S_n)| = 0.$$

By the above argument, at least one of the eigenvalues must be equal to zero. If $\lambda = 0$, then we get

$$\begin{aligned} |\lambda I - \mathrm{SM}^e(S_n)| &= |0I - \mathrm{SM}^e(S_n)| = |-\mathrm{SM}^e(S_n)|\\ &= (-1)^n|\mathrm{SM}^e(S_n)| = 0. \end{aligned}$$

Also

$$P^{se}_{S_n}(\lambda) = \lambda^{n-2}\left[\lambda^2 - n^2(n-1)\right].$$

In this equation $P^{se}_{S_n}(0) = 0$ so the result is verified.

If $\lambda \neq 0$, then we have the form of $P^{se}_{S_n}(\lambda)$ as

$$P^{se}_{S_n}(\lambda) = \begin{vmatrix} \lambda & -n & 0 & 0 & 0 & \cdots & 0 & 0\\ -n & \lambda & -n & -n & -n & \cdots & -n & -n\\ 0 & -n & \lambda & 0 & 0 & \cdots & 0 & 0\\ 0 & -n & 0 & 0 & 0 & 0 & 0 & 0\\ . & . & . & \cdots & . & . & . & .\\ 0 & -n & 0 & \cdots & 0 & 0 & \lambda & 0\\ 0 & -n & 0 & \cdots & 0 & 0 & 0 & \lambda \end{vmatrix}.$$

Proceeding as above, the result is obtained. ∎

Now we calculate the sum-edge characteristic polynomial of the complete graph K_n:

Theorem 2.5: *The formula for the sum-edge characteristic polynomial of the complete graph K_n obtained by means of the sum-edge matrix is*

$$P^{se}_{K_n}(\lambda) = \left[\lambda - 2(n-1)^2\right]\left[\lambda + 2(n-1)\right]^{n-1}.$$

Proof:

$$P^{se}_{K_n}(\lambda) = \begin{vmatrix} \lambda & -2(n-1) & -2(n-1) & \cdots & -2(n-1)\\ -2(n-1) & \lambda & -2(n-1) & \cdots & -2(n-1)\\ . & . & . & . & .\\ -2(n-1) & \cdots & -2(n-1) & \lambda & -2(n-1)\\ -2(n-1) & \cdots & -2(n-1) & -2(n-1) & \lambda \end{vmatrix}.$$

If we use the elementary row operation $R_1 + R_2 + \cdots + R_n \longrightarrow R_1$, we get

$$= \begin{vmatrix} \lambda - 2(n-1)^2 & \lambda - 2(n-1)^2 & \lambda - 2(n-1)^2 & \cdots & \lambda - 2(n-1)^2\\ -2(n-1) & \lambda & -2(n-1) & \cdots & -2(n-1)\\ . & . & . & . & .\\ -2(n-1) & \cdots & -2(n-1) & \lambda & -2(n-1)\\ -2(n-1) & \cdots & -2(n-1) & -2(n-1) & \lambda \end{vmatrix}$$

$$= \left[\lambda - 2(n-1)^2\right]\begin{vmatrix} 1 & 1 & 1 & \cdots & 1\\ -2(n-1) & \lambda & -2(n-1) & \cdots & -2(n-1)\\ . & . & . & . & .\\ -2(n-1) & \cdots & -2(n-1) & \lambda & -2(n-1)\\ -2(n-1) & \cdots & -2(n-1) & -2(n-1) & \lambda \end{vmatrix}.$$

If we use the elementary row operations

$$\begin{gathered} 2(n-1)R_1 + R_2 \longrightarrow R_2\\ 2(n-1)R_1 + R_3 \longrightarrow R_3\\ \cdots\\ 2(n-1)R_1 + R_n \longrightarrow R_n, \end{gathered}$$

we get

$$= \left[\lambda - 2(n-1)^2\right] \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \lambda + 2(n-1) & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & \lambda + 2(n-1) & 0 \\ 0 & \cdots & 0 & 0 & \lambda + 2(n-1) \end{vmatrix}$$

$$= \left[\lambda - 2(n-1)^2\right] [\lambda + 2(n-1)]^{n-1}.$$

■

Finally, we shall calculate the sum-edge characteristic polynomial of the complete bipartite graph $K_{r,s}$. We shall omit the proof as it is similar to the above cases.

Theorem 2.6: *The formula for the sum-edge characteristic polynomial of the complete bipartite graph $K_{r,s}$ obtained by means of the sum-edge matrix is*

$$P^{se}_{K_{r,s}}(\lambda) = \lambda^{r+s-2}\left[\lambda^2 - (r+s)^2 rs\right].$$

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