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Motional EMF/MMF in moving medium with magnetic monopoles

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We derived the motional emf/mmf (electromotive force/magnetomotive force) in moving medium (in a Galilean frame) with magnetic monopoles using two different formulations: one where the Lorentz fields are continuously differentiable vector functions and one where they are not required to be. We then compared the two formulations in physical meaning.

K e y w o r d s: motional emf/mmf, moving medium, Lorentz fields

1 Introduction

Since macroscopic field quantities cannot be measured in the medium, there are many theories of electromagnetism in a moving or stationary material medium (with magnetic monopoles or without).

Minkowski [1] introduced the theory of the electrodynamics of moving medium, which is based on the relativity principle. In Minkowski's theory, the forms of the Maxwell equations in free space are invariant for uniformly moving medium, but the constitutive equations are altered. Amperian current loop theory [2] and Chu theory [3] define the source terms in Maxwell equations in stationary medium, and these terms are modified for moving medium.

Stratton [4] gave motional emf/mmf equations in integral form in stationary medium and said that these equations can be generalized to moving medium. The generalization of the equations in stationary medium to moving medium requires special care. Tai [5] explained the field and source quantities and gave motional emf/mmf in moving medium, where the Lorentz fields are continuously differentiable vector functions. Rodrigues [6] studied the conditions for the equivalence of some different forms of Faraday's law from a mathematical perspective. Benedetto [7] gave the general formulation of the Maxwell Faraday equation where the Lorentz field is not required to be a continuously differentiable vector function.

The field quantities of Maxwell equations in stationary material medium with magnetic monopoles have been addressed by many authors. Artru and Fayolle [8] and then McDonald [9] gave the field quantities inside the medium with magnetic monopoles directly. Zor [10] used duality and dimensional analysis to construct the new field quantities in material medium with magnetic monopoles.

2 Formulation

The Maxwell equations in material medium with magnetic monopoles relating vector fields and scalar fields are as follows [10]:

$$c^2\nabla\times\boldsymbol{D}_m(\boldsymbol{r},t) = -\frac{\partial\boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t} - \boldsymbol{J}_{fm}(\boldsymbol{r},t), \tag{1}$$

$$\nabla\times\boldsymbol{H}_e(\boldsymbol{r},t) = \frac{\partial\boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t} + \boldsymbol{J}_{fe}(\boldsymbol{r},t), \tag{2}$$

$$\nabla\cdot\boldsymbol{D}_e(\boldsymbol{r},t) = \rho_{fe}(\boldsymbol{r},t), \tag{3}$$

$$\nabla\cdot\boldsymbol{H}_m(\boldsymbol{r},t) = \rho_{fm}(\boldsymbol{r},t), \tag{4}$$

with the following constitutive equations:

$$\boldsymbol{D}_e(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{P}_e(\boldsymbol{r},t), \tag{5}$$

$$\boldsymbol{H}_e(\boldsymbol{r},t) = \frac{\boldsymbol{B}(\boldsymbol{r},t)}{\mu_0} - \boldsymbol{M}_e(\boldsymbol{r},t), \tag{6}$$

$$\boldsymbol{D}_m(\boldsymbol{r},t) = \varepsilon_0\boldsymbol{E}(\boldsymbol{r},t) - \boldsymbol{P}_m(\boldsymbol{r},t), \tag{7}$$

$$\boldsymbol{H}_m(\boldsymbol{r},t) = \frac{\boldsymbol{B}(\boldsymbol{r},t)}{\mu_0} + \boldsymbol{M}_m(\boldsymbol{r},t)\,. \tag{8}$$

The Lorentz force on free charges and current densities in material medium can be defined in two sets of equations [10],

$$\boldsymbol{F}_e(\boldsymbol{r},t) = \frac{1}{\varepsilon_0}\rho_{fe}(\boldsymbol{r},t)\boldsymbol{D}_m(\boldsymbol{r},t) + \mu_0\boldsymbol{J}_{fe}(\boldsymbol{r},t)\times\boldsymbol{H}_m(\boldsymbol{r},t), \tag{9}$$

$$\boldsymbol{F}_m(\boldsymbol{r},t) = \mu_0\rho_{fm}(\boldsymbol{r},t)\boldsymbol{H}_e(\boldsymbol{r},t) - \mu_0\boldsymbol{J}_{fm}(\boldsymbol{r},t)\times\boldsymbol{D}_e(\boldsymbol{r},t). \tag{10}$$

Using equations (9) and (10) for the charges on a moving loop in a vector field, we can define new field quantities (effective electric and magnetic fields),

$$\boldsymbol{E}^*(\boldsymbol{r},t) = \frac{1}{\varepsilon_0}\boldsymbol{D}_m(\boldsymbol{r},t) + \mu_0\mathbf{v}(\boldsymbol{r},t)\times\boldsymbol{H}_m(\boldsymbol{r},t), \tag{11}$$

$$\boldsymbol{B}^*(\boldsymbol{r},t) = \mu_0\boldsymbol{H}_e(\boldsymbol{r},t) - \mu_0\mathbf{v}(\boldsymbol{r},t)\times\boldsymbol{D}_e(\boldsymbol{r},t). \tag{12}$$

The second terms on the right side of (11) and (12) can be called Lorentz fields. We assume that the fields $\boldsymbol{D}_e$, $\boldsymbol{H}_e$, $\boldsymbol{D}_m$, and $\boldsymbol{H}_m$ are continuous and have continuous space derivatives. We drive the motional emf/mmf using these effective fields both where the Lorentz fields are continuously differentiable vector functions and where they are not required to be.

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2.1 The formulation in which Lorentz fields are continuously differentiable vector functions

We consider a closed loop C that bounds a surface S moving with velocity $\boldsymbol{v}$ in a vector field. When $\boldsymbol{v}$ is a continuous field with continuous space derivatives, we obtain new integral forms of the Maxwell Faraday and Ampere equations.

We can express (1) by the equivalent integral relation,

$$c^2 \int_S \nabla \times \boldsymbol{D}_m(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S} = -\int_S \frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t} \cdot \mathrm{d}\boldsymbol{S} - \int_S \boldsymbol{J}_{fm}(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S}\,. \tag{13}$$

If we substitute the new field quantity (11) into (13), we obtain

$$\frac{1}{\mu_0} \int_S \nabla \times (\boldsymbol{E}^*(\boldsymbol{r},t) - \mu_0 \mathbf{v}(\boldsymbol{r},t) \times \boldsymbol{H}_m(\boldsymbol{r},t)) \cdot \mathrm{d}\boldsymbol{S} = -\int_S \frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t} \cdot d\boldsymbol{S} - \int_S \boldsymbol{J}_{fm}(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S}\,. \tag{14}$$

We can then apply the Stokes theorem and obtain

$$\frac{1}{\mu_0} \oint_C \boldsymbol{E}^*(\boldsymbol{r},t) \cdot d\boldsymbol{c} = -\int_S \Big[\frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t} + \boldsymbol{J}_{fm}(\boldsymbol{r},t) - \nabla \times (\mathbf{v}(\boldsymbol{r},t) \times \boldsymbol{H}_m(\boldsymbol{r},t))\Big] \cdot \mathrm{d}\boldsymbol{S}\,. \tag{15}$$

Considering (4), we can write (15) in the form

$$\frac{1}{\mu_0} \oint_C \boldsymbol{E}^*(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{c} = -\int_S \Big[\frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t} + (\nabla \cdot \boldsymbol{H}_m(\boldsymbol{r},t))\mathbf{v}(\boldsymbol{r},t) - \nabla \times (\mathbf{v}(\boldsymbol{r},t) \times \boldsymbol{H}_m(\boldsymbol{r},t)) + \boldsymbol{J}_{fm}(\boldsymbol{r},t) - \rho_{fm}(\boldsymbol{r},t)\mathbf{v}(\boldsymbol{r},t)\Big] \cdot \mathrm{d}\boldsymbol{S}\,. \tag{16}$$

Now, we can define a new magnetic current density,

$$\boldsymbol{J}^*{}_m(\boldsymbol{r},t) = \boldsymbol{J}_{fm}(\boldsymbol{r},t) - \rho_{fm}(\boldsymbol{r},t)\mathbf{v}(r,t). \tag{17}$$

If the field $\mathbf{v}$ is continuous with continuous space derivatives, we can apply the Helmholtz formula to (16). Hence, we obtain the general integral form of the Maxwell Faraday equation (motional emf),

$$\frac{1}{\mu_0} \oint_C \boldsymbol{E}^*(\boldsymbol{r},t) \cdot d\boldsymbol{c} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \boldsymbol{H}_m(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S} - \int_S \boldsymbol{J}^*_m(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S}\,. \tag{18}$$

We could apply the Helmholtz formula directly to (13) to obtain the general integral form (18), but we preferred the other method to yield (15). This equation can be applied to certain physical problems. We see that the connection between equations (15) and (18) is provided under the condition that the Lorentz field is a continuously differentiable vector function.

We can construct an analogous equation called the Maxwell Ampere equation (motional mmf) by applying a similar method. We can obtain the integral form of (2) in suitable conditions,

$$\int_S \nabla \times \boldsymbol{H}_e(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S} = \int_S \frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t} \cdot \mathrm{d}\boldsymbol{S} + \int_S \boldsymbol{J}_{fe}(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S}\,. \tag{19}$$

If we substitute the new field quantity (12) into (19) and apply the Stokes theorem, it yields

$$\frac{1}{\mu_0} \oint_C \boldsymbol{B}^*(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{c} = \int_S \Big[\frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t} + \boldsymbol{J}_{fe}(\boldsymbol{r},t) - \nabla \times (\mathbf{v}(\boldsymbol{r},t) \times \boldsymbol{D}_e(\boldsymbol{r},t))\Big] \cdot \mathrm{d}\boldsymbol{S}\,. \tag{20}$$

Considering (3), we can write (20) in the form

$$\frac{1}{\mu_0} \oint_C \boldsymbol{B}^*(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{c} = \int_S \Big[\frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t} + \big(\nabla \cdot \boldsymbol{D}_e(\boldsymbol{r},t)\mathbf{v}(\boldsymbol{r},t)\big) - \nabla \times \big(\mathbf{v}(r,t) \times \boldsymbol{D}_e(\boldsymbol{r},t)\big) + \boldsymbol{J}_{fe}(\boldsymbol{r},t) - \rho_{fe}(\boldsymbol{r},t)\mathbf{v}(r,t)\Big] \cdot \mathrm{d}\boldsymbol{S}\,. \tag{21}$$

We can then define a new electric current density,

$$\boldsymbol{J}^*_e(\boldsymbol{r},t) = \boldsymbol{J}_{fe}(\boldsymbol{r},t) - \rho_{fe}(\boldsymbol{r},t)\mathbf{v}(\boldsymbol{r},t)\,. \tag{22}$$

We can apply the Helmholtz formula to right side of (21) and obtain the general integral form of the motional mmf,

$$\frac{1}{\mu_0} \oint_C \boldsymbol{B}^*(\boldsymbol{r},t) \cdot d\boldsymbol{c} = \frac{\mathrm{d}}{\mathrm{d}t} \int_S \frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t} \cdot \mathrm{d}\boldsymbol{S} + \int_S \boldsymbol{J}^*_e(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S}. \tag{23}$$

Note that equations (18) and (23) are general forms, but equations (15) and (20) are applied to physical problems under conditions where the Lorentz fields are continuously differentiable vector functions.

2.2 The formulation in which Lorentz fields are not required to be continuously differentiable vector functions

Now, we construct a new formulation for general forms of the motional emf and mmf ((18), (23)) in which the Lorentz fields are not required to be continuously differentiable vector functions.

We consider a moving closed loop C that bounds a surface S in a vector field. In this method, the velocity $\boldsymbol{v}$ of the differential element $\mathrm{d}\boldsymbol{c}$ along C is not required to be uniform. If the surface bounded by the loop moves in time interval t_i and $t_f = t_i + \mathrm{d}t$, it sweeps the imaginary volume ϑ. This imaginary volume is enclosed by the surfaces $S(t_i)$, $S(t_f)$ and lateral surface $\mathrm{d}\Sigma$. In this time interval, the lateral differential surface element is equal to $\mathrm{d}\boldsymbol{A} = \mathrm{d}\boldsymbol{c} \times \mathrm{d}\ell$ where $\mathrm{d}\boldsymbol{c}$ moves by $\mathrm{d}\ell = \mathbf{v}\mathrm{d}t$.

When $\boldsymbol{v}$ is not required to be a continuous field with continuous space derivatives, we can begin with the general equation (18). The second term of (18) is the rate of change of the flux ϕ produced by the continuous vector field $\boldsymbol{H}_m$ with continuous derivatives across the moving surface S. This term can be written as

$$\frac{\mathrm{d}\phi_m(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{1}{\mathrm{d}t}\Big[\int\limits_{S(t+\mathrm{d}t)} \boldsymbol{H}_m(\boldsymbol{r},t+\mathrm{d}t)\cdot\mathrm{d}\boldsymbol{S} - \int\limits_{S(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S}\Big]. \tag{24}$$

We can find the value of $\boldsymbol{H}_m(\boldsymbol{r}, t+dt)$ in terms of $\boldsymbol{H}_m(\boldsymbol{r},t)$ using Taylor's theorem,

$$\boldsymbol{H}_m(\boldsymbol{r},t+\mathrm{d}t) = \boldsymbol{H}_m(\boldsymbol{r},t) + \frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t}\mathrm{d}t. \tag{25}$$

If we substitute (25) into (24), we obtain

$$\frac{\mathrm{d}\phi_m(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{1}{\mathrm{d}t}\Big[\int\limits_{S(t+\mathrm{d}t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} - \int\limits_{S(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S}\Big] + \int\limits_{S(t+\mathrm{d}t)} \frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t}\cdot\mathrm{d}\boldsymbol{S}. \tag{26}$$

The second term in the right side of (26) becomes, in the limit,

$$\lim_{\mathrm{d}t\to 0} \int\limits_{S(t+\mathrm{d}t)} \frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t}\cdot\mathrm{d}\boldsymbol{S} = \int\limits_{S(t)} \frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t}\cdot\mathrm{d}\boldsymbol{S}. \tag{27}$$

Thus, this term represents the variation due only to the change over time of $\boldsymbol{H}_m$. If we substitute (27) into (26), we obtain

$$\frac{\mathrm{d}\phi_m(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{1}{\mathrm{d}t}\Big[\int\limits_{S(t+\mathrm{d}t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} - \int\limits_{S(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S}\Big] + \int\limits_{S(t)} \frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t}\cdot\mathrm{d}\boldsymbol{S}. \tag{28}$$

The first term of the right side of (28) is the total derivative of the flux caused by the vector field $\boldsymbol{H}_m$ when constant in time. We consider this expression by applying the Gauss theorem on the imaginary volume produced by moving closed loop C,

$$\int\limits_{\vartheta(t)} \nabla\cdot\boldsymbol{H}_m(\boldsymbol{r},t)\mathrm{d}\vartheta = \int\limits_{S(t+\mathrm{d}t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} - \int\limits_{S(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} + \int\limits_{\mathrm{d}\Sigma(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{A}. \tag{29}$$

We can write the flux expression as

$$\mathrm{d}\phi_{cm}(\boldsymbol{r},t) = \int\limits_{S(t+\mathrm{d}t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} - \int\limits_{S(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S} = \int\limits_{\vartheta(t)} \nabla\cdot\boldsymbol{H}_m(\boldsymbol{r},t)\mathrm{d}\vartheta - \int\limits_{\mathrm{d}\Sigma(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{A}. \tag{30}$$

The volume element and the surface element of the lateral surface are defined as

$$\begin{aligned} \mathrm{d}\vartheta &= \mathrm{d}\boldsymbol{S}\cdot\mathrm{d}\ell = \mathrm{d}\boldsymbol{S}\cdot\mathbf{v}(\boldsymbol{r},t)\mathrm{d}t, \\ \mathrm{d}\boldsymbol{A} &= \mathrm{d}\boldsymbol{c}\times\mathrm{d}\ell = \mathrm{d}\boldsymbol{c}\times\mathbf{v}(\boldsymbol{r},t)\mathrm{d}t. \end{aligned} \tag{31}$$

Thus, we can write

$$\mathrm{d}\phi_{cm}(\boldsymbol{r},t) = \int\limits_{S(t)} \big(\nabla\cdot\boldsymbol{H}_m(\boldsymbol{r},t)\big)\mathbf{v}(\boldsymbol{r},t)\cdot\mathrm{d}\boldsymbol{S}\,\mathrm{d}t - \oint\limits_{C(t)} \boldsymbol{H}_m(\boldsymbol{r},t)\cdot\big(\mathrm{d}\boldsymbol{c}\times\mathbf{v}(\boldsymbol{r},t)\big)\mathrm{d}t. \tag{32}$$

Using the vector identity, we can then write

$$\boldsymbol{H}_m(\boldsymbol{r},t)\cdot\big(\mathrm{d}\boldsymbol{c}\times\mathbf{v}(\boldsymbol{r},t)\big) = \big(\mathbf{v}(\boldsymbol{r},t)\times\boldsymbol{H}_m(\boldsymbol{r},t)\big)\cdot\mathrm{d}\boldsymbol{c}, \tag{33}$$

and obtain

$$\frac{\mathrm{d}\phi_{cm}(\boldsymbol{r},t)}{\mathrm{d}t} = \int\limits_{S(t)} \big(\nabla\cdot\boldsymbol{H}_m(\boldsymbol{r},t)\big)\boldsymbol{v}(r,t)\cdot\mathrm{d}\boldsymbol{S} - \oint\limits_{C(t)} \big(\mathbf{v}(\boldsymbol{r},t)\times\boldsymbol{H}_m(\boldsymbol{r},t)\big)\cdot\mathrm{d}\boldsymbol{c}. \tag{34}$$

If we substitute (34) into (28), we obtain

$$\frac{\mathrm{d}\phi_m(\boldsymbol{r},t)}{\mathrm{d}t} = \int\limits_{S(t)} \Big[\frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t} + \big(\nabla\cdot\boldsymbol{H}_m(\boldsymbol{r},t)\big)\boldsymbol{v}(\boldsymbol{r},t)\Big]\cdot\mathrm{d}\boldsymbol{S} - \oint\limits_{C(t)} \big(\mathbf{v}(\boldsymbol{r},t)\times\boldsymbol{H}_m(\boldsymbol{r},t)\big)\cdot\mathrm{d}\boldsymbol{c}. \tag{35}$$

And if we substitute (35) into (18), we obtain the equation of motional emf,

$$\frac{1}{\mu_0}\oint_{C(t)} \boldsymbol{E}^*(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{c} = -\int_{S(t)} \Big[\frac{\partial \boldsymbol{H}_m(\boldsymbol{r},t)}{\partial t} + \big(\nabla\cdot \boldsymbol{H}_m(\boldsymbol{r},t)\big)\boldsymbol{v}(\boldsymbol{r},t)\Big]\cdot \mathrm{d}\boldsymbol{S} + \oint_{C(t)} \big(\boldsymbol{v}(\boldsymbol{r},t)\times \boldsymbol{H}_m(\boldsymbol{r},t)\big)\cdot \mathrm{d}\boldsymbol{c} - \int_{S(t)} \boldsymbol{J}_m^*(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S}\,. \tag{36}$$

If $\boldsymbol{v}$ were been continuous with continuous space derivatives, we could apply Stokes theorem to (36) and obtain the same equation as (16).

We begin with the general equation (23) to obtain the motional mmf using a similar method. The second term of (23) can be written as

$$\frac{\mathrm{d}\phi_e(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{1}{\mathrm{d}t}\Big[\int_{S(t+\mathrm{d}t)} \boldsymbol{D}_e(\boldsymbol{r},t+\mathrm{d}t)\cdot \mathrm{d}\boldsymbol{S} - \int_{S(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S}\Big]. \tag{37}$$

Using Taylor's theorem, we obtain

$$\frac{\mathrm{d}\phi_e(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{1}{\mathrm{d}t}\Big[\int_{S(t+\mathrm{d}t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S} - \int_{S(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S}\Big] + \int_{S(t+\mathrm{d}t)} \frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t}\cdot \mathrm{d}\boldsymbol{S}\,. \tag{38}$$

The second term on the right side of (38) becomes, in the limit,

$$\lim_{\mathrm{d}t\to 0}\int_{S(t+\mathrm{d}t)} \frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t}\cdot \mathrm{d}\boldsymbol{S} = \int_{S(t)} \frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t}\cdot \mathrm{d}\boldsymbol{S}\,. \tag{39}$$

If we substitute (39) into (38), we obtain

$$\frac{\mathrm{d}\phi_e(\boldsymbol{r},t)}{\mathrm{d}t} = \frac{1}{\mathrm{d}t}\Big[\int_{S(t+\mathrm{d}t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S} - \int_{S(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S}\Big] + \int_{S(t)} \frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t}\cdot \mathrm{d}\boldsymbol{S}\,. \tag{40}$$

We consider the second term of (40), applying the Gauss theorem to the imaginary volume produced by moving the closed loop C,

$$\int_{\vartheta(t)} \nabla\cdot \boldsymbol{D}_e(\boldsymbol{r},t)\mathrm{d}\vartheta = \int_{S(t+\mathrm{d}t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S} - \int_{S(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S} + \int_{d\Sigma(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{A}\,. \tag{41}$$

And the flux expression can be written as

$$\mathrm{d}\phi_{ce}(\boldsymbol{r},t) = \int_{S(t+\mathrm{d}t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S} - \int_{S(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S} = \int_{\vartheta(t)} \nabla\cdot \boldsymbol{D}_e(\boldsymbol{r},t)\mathrm{d}\vartheta - \int_{\mathrm{d}\Sigma(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{A}\,. \tag{42}$$

Using the definitions of volume and surface elements given in (31), we can write

$$\mathrm{d}\phi_{ce}(\boldsymbol{r},t) = \int_{S(t)} \big(\nabla\cdot \boldsymbol{D}_e(\boldsymbol{r},t)\big)\boldsymbol{v}(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S}\mathrm{d}t - \oint_{C(t)} \boldsymbol{D}_e(\boldsymbol{r},t)\cdot\big(\mathrm{d}\boldsymbol{c}\times \boldsymbol{v}(\boldsymbol{r},t)\big)\mathrm{d}t\,. \tag{43}$$

Then, using the vector identity, we can write

$$\boldsymbol{D}_e(\boldsymbol{r},t)\cdot\big(\mathrm{d}\boldsymbol{c}\times \boldsymbol{v}(\boldsymbol{r},t)\big) = \big(\boldsymbol{v}(\boldsymbol{r},t)\times \boldsymbol{D}_e(\boldsymbol{r},t)\big)\cdot \mathrm{d}\boldsymbol{c}\,, \tag{44}$$

and obtain

$$\frac{\mathrm{d}\phi_{ce}(\boldsymbol{r},t)}{\mathrm{d}t} = \int_{S(t)} \big(\nabla\cdot \boldsymbol{D}_e(\boldsymbol{r},t)\big)\boldsymbol{v}(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S} - \oint_{C(t)} \big(\boldsymbol{v}(\boldsymbol{r},t)\times \boldsymbol{D}_e(\boldsymbol{r},t)\big)\cdot \mathrm{d}\boldsymbol{c}\,. \tag{45}$$

If we substitute (45) into (40), we obtain

$$\frac{\mathrm{d}\phi_e(\boldsymbol{r},t)}{\mathrm{d}t} = \int_{S(t)} \Big[\frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t} + \big(\nabla\cdot \boldsymbol{D}_e(\boldsymbol{r},t)\big)\boldsymbol{v}(\boldsymbol{r},t)\Big]\cdot \mathrm{d}\boldsymbol{S} - \oint_{C(t)} \big(\boldsymbol{v}(\boldsymbol{r},t)\times \boldsymbol{D}_e(\boldsymbol{r},t)\big)\cdot \mathrm{d}\boldsymbol{c}\,. \tag{46}$$

And if we substitute (46) into (23), we obtain the equation of motional mmf,

$$\frac{1}{\mu_0}\oint_{C(t)} \boldsymbol{B}^*(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{c} = \int_{S(t)} \Big[\frac{\partial \boldsymbol{D}_e(\boldsymbol{r},t)}{\partial t} + \big(\nabla\cdot \boldsymbol{D}_e(\boldsymbol{r},t)\big)\boldsymbol{v}(\boldsymbol{r},t)\Big]\cdot \mathrm{d}\boldsymbol{S} - \oint_{C(t)} \big(\boldsymbol{v}(\boldsymbol{r},t)\times \boldsymbol{D}_e(\boldsymbol{r},t)\big)\cdot \mathrm{d}\boldsymbol{c} + \int_{S(t)} \boldsymbol{J}_e^*(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{S}\,. \tag{47}$$

Similarly, if $\boldsymbol{v}$ were continuous with continuous space derivatives, we could apply the Stokes theorem and obtain the same equation as (21).

The motional emf/mmf equations (36), (47) can explain physical problems including discontinuous velocity functions. However, we can use equations (16) and (21) under the condition that Lorentz fields are continuously differentiable vector functions.

3 Conclusion

We constructed the motional emf/mmf in moving medium (in a Galilean frame) with magnetic monopoles using two formulations (one in which Lorentz fields are continuously differentiable vector functions and one in which they are not required to be). The general case is the method using Lorentz fields that are not required to be continuously differentiable vector functions, since this method explains all the motions of the medium.

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